

A CORRECTION TO THE HYDRODYNAMIC LIMIT OF BOUNDARY DRIVEN EXCLUSION PROCESSES IN A SUPER-DIFFUSIVE TIME SCALE

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ABSTRACT. We consider a one-dimensional, weakly asymmetric, boundary driven exclusion process on the interval $[0, N] \cap \mathbb{Z}$ in the super-diffusive time scale $N^2 \epsilon_N^{-1}$, where $1 \ll \epsilon_N^{-1} \ll N^{1/4}$. We assume that the external field and the chemical potentials, which fix the density at the boundaries, evolve smoothly in the macroscopic time scale. We derive an equation which describes the evolution of the density up to the order ϵ_N .

1. INTRODUCTION

A theory of thermodynamic transformations for nonequilibrium stationary states has been proposed recently [4, 5] in the framework of the Macroscopic Fluctuation Theory [1, 2]. It defined the renormalized work performed by a transformation between two nonequilibrium stationary states in driven diffusive systems, and it proved a Clausius inequality which postulates that the renormalized work is always larger than the variation of the equilibrium free energy between the final and the initial nonequilibrium states.

In quasi-static transformations, transformations in which the variations of the environment are very slow, the renormalized work coincides asymptotically with the variation of the equilibrium free energy. More precisely, fix a transformation $u(t)$, $t \geq 0$, between two nonequilibrium stationary states, and denote by $W^{\text{ren}}(u)$ the renormalized work performed by u . Let u_ϵ be the transformation u slowed down by a parameter $\epsilon > 0$, $u_\epsilon(t) = u(t\epsilon)$. Then, $\lim_{\epsilon \rightarrow 0} W^{\text{ren}}(u_\epsilon) = \Delta F$, where ΔF represents the variation of the equilibrium free energy between the final and the initial nonequilibrium states. Note that the asymptotic identity is attained independently of the transformation u chosen.

Let us mention that the theory of thermodynamic transformations between nonequilibrium states, and the analysis of quasi-static transformations has been extended to the framework of stochastic perturbations of microscopic Hamiltonian dynamics in contact with heat baths in [13, 15, 14].

To select, among the slow transformations between two nonequilibrium stationary states, the one which minimizes the renormalized work we have to examine the first order term in the expansion in ϵ of the renormalized work. This question has been addressed in [3], where it was shown that for slow transformations between two equilibrium states the first order correction of the renormalized work is minimized by transformations whose intermediate states are equilibrium states, and

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where a partial differential equation which describes the evolution of the optimal transformation has been derived.

A time-change permits to convert a slow transformation in an ordinary transformation whose differential operator is multiplied by ϵ^{-1} . This observation brings us to the question of the correction to the hydrodynamic equation of boundary driven interacting particle systems.

Consider a symmetric, one-dimensional dynamics in contact with reservoirs and in the presence of an external field. At the macroscopic level the system is described by a local density $\rho(t, x)$, $x \in [0, 1]$, which evolves according to the driven diffusive equation

$$\begin{cases} \partial_t \rho = \partial_x(D(\rho)\partial_x \rho) - \partial_x(\chi(\rho)E) \\ f'(\rho(t, a)) = \lambda_a(t) \quad \text{for } a = 0, 1, \end{cases} \quad (1.1)$$

where D is the diffusivity, χ the mobility, $E(t, x)$ an external field, $\lambda_0(t)$, $\lambda_1(t)$ time-dependent chemical potentials, which fix the density at the boundaries, and f the equilibrium free energy density.

For a fixed external field $E(x)$ and a chemical potential $\lambda = (\lambda_0, \lambda_1)$, denote by $\bar{\rho}_{\lambda, E}$ the solution of the elliptic equation

$$\begin{cases} \partial_x(D(\rho)\partial_x \rho) - \partial_x(\chi(\rho)E) = 0, \\ f'(\rho(a)) = \lambda_a \quad \text{for } a = 0, 1. \end{cases} \quad (1.2)$$

Consider the driven diffusive equation (1.1) speeded up by ϵ^{-1} . Fix a transformation $(\lambda(t), E(t))$, $\epsilon > 0$, and a bounded profile $v_0 : [0, 1] \rightarrow \mathbb{R}$. Denote by $\rho_\epsilon(t)$ the solution of

$$\begin{cases} \partial_t \rho = \epsilon^{-1} \{ \partial_x(D(\rho)\partial_x \rho) - \partial_x(\chi(\rho)E(t)) \}, \\ f'(\rho(t, 0)) = \lambda_0(t), \quad f'(\rho(t, 1)) = \lambda_1(t), \\ \rho(0, \cdot) = \bar{\rho}_{\lambda(0), E(0)}(\cdot) + \epsilon v_0(\cdot). \end{cases} \quad (1.3)$$

A formal expansion in ϵ yields that, for $t > 0$, $u_\epsilon(t) = \epsilon^{-1}[\rho_\epsilon(t) - \bar{\rho}_{\lambda(t), E(t)}]$ converges to $v(t)$, the solution of the elliptic equation

$$\begin{cases} \partial_t \bar{\rho}_{\lambda(t), E(t)} = \partial_x^2(D(\bar{\rho}_{\lambda(t), E(t)})v) - \partial_x(\chi'(\bar{\rho}_{\lambda(t), E(t)})E(t)v), \\ v(0) = v(1) = 0. \end{cases} \quad (1.4)$$

Note that the limit v_t does not depend on the initial condition v_0 .

The main results of this article, Theorems 2.4 and 3.1, state a similar result for a microscopic dynamics speeded-up super-diffusively. Consider a one-dimensional, weakly asymmetric, exclusion process evolving on $\{1, \dots, N-1\}$, and in contact with reservoirs at the boundaries. Assume that the density of each reservoir evolves smoothly in the macroscopic time-scale, and that the dynamics is speeded-up by $N^2 \epsilon_N^{-1}$, where $\epsilon_N \rightarrow 0$ as $N \uparrow \infty$. De Masi and Olla [12] proved that starting from any initial distribution, at all macroscopic time $t > 0$ the system converges to a local equilibrium state whose density profile is given by the solution of the elliptic equation (1.2) with chemical potential $\lambda(t)$.

We examine in this article the correction to the hydrodynamic equation. Assume that $\epsilon_N^4 N \rightarrow \infty$, and that the exclusion process starts from a local equilibrium state associated to the density profile $\bar{\rho}_{\lambda(0), E(0)} + \epsilon_N v_0$. Then, for all $t > 0$, the system remains close, in the scale ϵ_N^{-1} , to a local equilibrium state whose density profile is given by $\bar{\rho}_{\lambda(t), E(t)} + \epsilon_N v_t$, where v_t is the solution of the elliptic equation

(1.4). More precisely, for every cylinder function Ψ , and every continuous function $H : [0, 1] \rightarrow \mathbb{R}$, if η_t^N represents the state at time t of the speeded-up exclusion process,

$$\frac{1}{\epsilon_N N} \sum_{j=1}^{N-1} H(j/N) \{ \tau_j \Psi(\eta_t^N) - E_{\bar{\rho}_{\lambda(t), E(t) + \epsilon_N v_t}}[\Psi] \} \rightarrow 0 .$$

In this formula, $\{\tau_j : j \in \mathbb{Z}\}$ represents the group of translations and E_γ the expectation with respect to the local equilibrium state associated to the density profile γ .

The proof of the main results follows the strategy proposed by [16, 6], which consists in estimating the relative entropy of the state of the process with respect to the local equilibrium state whose density profile solves equation (1.3) with $\epsilon = \epsilon_N$. If $H_N(t)$ represents this latter relative entropy, the main result asserts that for all $t > 0$,

$$\frac{1}{N \epsilon_N^2} H_N(t) \rightarrow 0 .$$

The results presented in this article have a similarity to the correction to the hydrodynamic equation, examined in [6, 10] in the asymmetric case in dimension $d \geq 3$ and in [7] in the symmetric case.

2. NOTATION AND MAIN RESULTS

2.1. The model. We examine a one-dimensional weakly asymmetric exclusion process in contact with reservoirs. Fix $\Lambda = (0, 1)$, and let $\Lambda_N = \{1, \dots, N-1\}$, $N \geq 1$, be a discretization of Λ . The microscopic point $j \in \Lambda_N$ thus represents the macroscopic location $j/N \in \Lambda$. Particles evolve on Λ_N under an exclusion rule which allows at most one particle per site. The state space is denoted by $\Sigma_N = \{0, 1\}^{\Lambda_N}$, and the configurations are represented by the Greek letters η, ξ so that $\eta(j) = 1$ if site $j \in \Lambda_N$ is occupied for the configuration η , and $\eta(j) = 0$ otherwise.

Let A_0 be a finite subset of \mathbb{Z} which contains the set $\{0, 1\}$. Consider a strictly positive function $c : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}_+$ which does not depend on the variables $\eta(0)$ and $\eta(1)$ and whose support is contained in A_0 :

$$c(\eta) = c_\emptyset + \sum_{\substack{A \subset A_0 \\ A \cap \{0, 1\} = \emptyset}} c_A \prod_{k \in A} \eta(k) ,$$

where c_A are coefficients which may be negative. In the case where $A_0 = \{0, 1\}$, $c(\eta)$ is constant equal to c_\emptyset .

Denote by $\{\tau_k : k \in \mathbb{Z}\}$ the group of translations in $\{0, 1\}^{\mathbb{Z}}$ so that $\tau_k \eta$ is the configuration defined by $(\tau_k \eta)(j) = \eta(k + j)$, $k, j \in \mathbb{Z}$. The action is extended to cylinder functions $\Psi : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$, in the usual way: $(\tau_k \Psi)(\eta) = \Psi(\tau_k \eta)$.

We assume throughout this article that the jump rate c satisfies the *gradient* condition: There exist $m \geq 1$, cylinder functions h_1, \dots, h_m , and finite-range, signed measures μ_1, \dots, μ_m on \mathbb{Z} with vanishing total mass such that

$$[\eta(0) - \eta(1)] c(\eta) = \sum_{a=1}^m \sum_{j \in \mathbb{Z}} \mu_a(j) (\tau_{-j} h_a)(\eta) . \quad (2.1)$$

This decomposition is clearly not unique. In the case $c(\eta) = 1 + \eta(-1) + \eta(2)$, one may take $m = 3$, $h_1(\eta) = \eta(-1)\eta(0)$, $h_2(\eta) = \eta(0)\eta(2)$, $h_3(\eta) = \eta(0)$, $\mu_1(0) = 1 = -\mu_1(2)$, $\mu_2(0) = 1 = -\mu_2(-1)$, $\mu_3(0) = 1 = -\mu_3(-1)$.

Fix a chemical potential $\lambda : \partial\Lambda \rightarrow \mathbb{R}$, where $\partial\Lambda$ represents the boundary of Λ . In one dimension, λ is simply a pair (λ_0, λ_1) . Let $\alpha = (\alpha_0, \alpha_1)$ be the density of particles associated to the chemical potential λ :

$$\alpha_0 = \frac{e^{\lambda_0}}{1 + e^{\lambda_0}}, \quad \alpha_1 = \frac{e^{\lambda_1}}{1 + e^{\lambda_1}}.$$

Let $\tau_j^{N,\lambda} : \Sigma_N \rightarrow \{\alpha_0, \alpha_1, 0, 1\}^{\mathbb{Z}}$, $N \geq 1$, $j \in \mathbb{Z}$, be the operators defined by

$$(\tau_j^{N,\lambda}\eta)(k) = \eta(k+j) \text{ if } k+j \in \Lambda_N, \quad (\tau_j^{N,\lambda}\eta)(k) = \begin{cases} \alpha_0 & \text{if } k+j \leq 0, \\ \alpha_1 & \text{if } k+j \geq N, \end{cases}$$

for $k \in \mathbb{Z}$. As before the action of the operator $\tau_j^{N,\lambda}$ can be extended to functions defined on Σ_N . For $N \geq 1$, $1 \leq j < N-1$, let the functions $c_{j,j+1}^{N,\lambda} : \Sigma_N \rightarrow \mathbb{R}_+$ be given by

$$c_{j,j+1}^{N,\lambda} = \tau_j^{N,\lambda} c,$$

so that $c_{j,j+1}^{N,\lambda}(\eta) = c(\tau_j^{N,\lambda}\eta)$. Note that $c_{0,1}^{N,\lambda}$ is usually not equal to c . It follows from (2.1) that for $N \geq 1$, $1 \leq j < N-1$,

$$w_{j,j+1}^{N,\lambda} := [\eta(j) - \eta(j+1)] c_{j,j+1}^{N,\lambda}(\eta) = \sum_{a=1}^m \sum_{k \in \mathbb{Z}} \mu_a(k) (\tau_{j-k}^{N,\lambda} h_a)(\eta). \quad (2.2)$$

We are now in a position to define the jump rates of the boundary driven exclusion process. Fix a smooth *external field* $E : [0, 1] \rightarrow \mathbb{R}$, and let

$$\begin{aligned} c_{0,1}^{N,\lambda,E}(\eta) &= r_{0,1}^\lambda(\eta) e^{(1/2N) E(0) [1-2\eta(1)]} c_{0,1}^{N,\lambda}(\eta), \\ c_{j,j+1}^{N,\lambda,E}(\eta) &= e^{(1/2N) E(j/N) [\eta(j) - \eta(j+1)]} c_{j,j+1}^{N,\lambda}(\eta), \quad 1 \leq j \leq N-2, \\ c_{N-1,N}^{N,\lambda,E}(\eta) &= r_{N-1,N}^\lambda(\eta) e^{(-1/2N) E(1) [1-2\eta(N-1)]} c_{N-1,N}^{N,\lambda}(\eta), \end{aligned}$$

where,

$$\begin{aligned} r_{0,1}^\lambda(\eta) &= \alpha_0[1 - \eta(1)] + \eta(1)[1 - \alpha_0], \\ r_{N-1,N}^\lambda(\eta) &= \alpha_1[1 - \eta(N-1)] + \eta(N-1)[1 - \alpha_1]. \end{aligned}$$

Denote by $L_N^{\lambda,E} = L_N$ the generator whose action on functions $f : \Sigma_N \rightarrow \mathbb{R}$ is given by

$$(L_N f)(\eta) = \sum_{j=0}^{N-1} c_{j,j+1}^{N,\lambda,E}(\eta) \{f(\sigma^{j,j+1}\eta) - f(\eta)\}. \quad (2.3)$$

In this formula, the configuration $\sigma^{j,j+1}\eta$, $1 \leq j \leq N-2$, represents the configuration obtained from η by exchanging the occupation variables $\eta(j)$, $\eta(j+1)$,

$$(\sigma^{j,j+1}\eta)(k) = \begin{cases} \eta(j+1), & k = j, \\ \eta(j), & k = j+1, \\ \eta(k), & k \neq j, j+1, \end{cases}$$

while $\sigma^{0,1}\eta$, $\sigma^{N-1,N}\eta$ represent the configuration obtained from η by flipping the occupation variables $\eta(1)$, $\eta(N-1)$, respectively:

$$(\sigma^j\eta)(k) = \begin{cases} \eta(k), & k \neq j, \\ 1 - \eta(k), & k = j. \end{cases}$$

2.2. Transformations. The dynamics introduced in the previous subsection is a finite-state, irreducible, continuous-time Markov chain. It has therefore a unique stationary state, denoted by $\nu_{\lambda,E}^N$. If the external field $E(x)$ vanishes and the chemical potentials coincide, $\lambda_0 = \lambda_1 = \lambda$, this stationary state is the Bernoulli product measure with density $\rho = e^\lambda/(1 + e^\lambda)$.

For a given continuous density profile $\gamma : [0, 1] \rightarrow [0, 1]$, Denote by $\nu_{\gamma(\cdot)}^N$ the product measure on Σ_N with marginals given by

$$\nu_{\gamma(\cdot)}^N \{ \eta(j) = 1 \} = \gamma(j/N), \quad j \in \Lambda_N. \quad (2.4)$$

Similarly, for $0 \leq \theta \leq 1$, ν_θ , stands for the Bernoulli product on $\{0, 1\}^{\mathbb{Z}}$ with density θ :

$$\nu_\theta \{ \eta(j) = 1 \} = \theta, \quad j \in \mathbb{Z}.$$

To describe the macroscopic evolution of the density, denote the *diffusivity* by $D : [0, 1] \rightarrow \mathbb{R}_+$, and the *mobility* by $\chi : [0, 1] \rightarrow \mathbb{R}_+$:

$$D(\theta) = E_{\nu_\theta}[c(\eta)], \quad \chi(\theta) = \frac{1}{2} E_{\nu_\theta}[\eta(1) - \eta(0)]^2 c(\eta) = \theta(1 - \theta) E_{\nu_\theta}[c(\eta)]. \quad (2.5)$$

The transport coefficients D and χ are related through the local Einstein relation

$$D(\theta) = \chi(\theta) f''(\theta), \quad (2.6)$$

where $f : [0, 1] \rightarrow \mathbb{R}$ the equilibrium free energy:

$$f(\theta) = \theta \log \theta + [1 - \theta] \log(1 - \theta).$$

Let $A = [0, 1]$ or \mathbb{R}_+ . Denote by $C^r(A)$, $r \geq 0$, the set of functions $F : A \rightarrow \mathbb{R}$ which are $[r]$ -times differentiable, where $[r]$ stands for the integer part of r , and whose $[r]$ -th derivative is Hölder continuous with exponent $r - [r]$, and by $C_0^r([0, 1])$ the set of functions in $C^r([0, 1])$ which vanish at the boundary. If r is an integer, we require the $[r]$ -th derivative to be continuous. Similarly, $C^{r,s}(\mathbb{R}_+ \times [0, 1])$, $r, s \geq 0$, represents the set of functions $F : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ which are $[r]$ -times differentiable in the time variable, $[s]$ -times differentiable in the space variable and whose $[r]$ -th (resp. $[s]$ -th) time (resp. space) derivative is Hölder continuous with exponent $r - [r]$ (resp. $s - [s]$). As before, if r or s is an integer, we require the corresponding derivative to be continuous.

Assume that $\lambda_a : \mathbb{R}_+ \rightarrow \mathbb{R}$, $a = 1, 2$, are functions in $C^1(\mathbb{R}_+)$ and that $E : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ is a function in $C^{1,2}(\mathbb{R}_+ \times [0, 1])$. Fix a density profile $\gamma : [0, 1] \rightarrow (0, 1)$ in $C^2([0, 1])$ and assume that there exists a function ψ in $C^{1+\beta/2, 2+\beta}(\mathbb{R}_+ \times [0, 1])$, for some $\beta > 0$, such that $f'(\psi(t, a)) = \lambda_a(t)$ for $a = 0, 1$, $\psi(0, x) = \gamma(x)$ for $x \in [0, 1]$, and such that

$$\partial_t \psi = \partial_x(D(\psi)\partial_x \psi) - \partial_x(\chi(\psi)E(t)) \quad \text{at } (t, x) = (0, 0) \text{ and } (t, x) = (0, 1).$$

Denote by $\rho(t, \cdot)$ the unique classical solution of the parabolic equation

$$\begin{cases} \partial_t \rho = \partial_x(D(\rho)\partial_x \rho) - \partial_x(\chi(\rho)E(t)), \\ f'(\rho(t, 0)) = \lambda_0(t), \quad f'(\rho(t, 1)) = \lambda_1(t), \\ \rho(0, \cdot) = \gamma(\cdot). \end{cases} \quad (2.7)$$

We refer to Theorem 6.1 of Chapter V in [11] for the existence and the uniqueness of classical solutions of equation (2.7).

Denote by $\mathcal{M}_N = \mathcal{M}(\Sigma_N)$ the set of probability measures on Σ_N endowed with the weak topology. For two probability measures μ, π in \mathcal{M}_N , let $H_N(\mu|\pi)$ be the relative entropy of μ with respect to π :

$$H_N(\mu|\pi) = \sup_f \left\{ \int f d\mu - \log \int e^f d\pi \right\},$$

where the supremum is carried over all functions $f : \Sigma_N \rightarrow \mathbb{R}$. It is well known [8] that the relative entropy has an explicit expression:

$$H_N(\mu|\pi) = \begin{cases} \int \log \frac{d\mu}{d\pi} d\mu & \text{if } \mu \ll \pi, \\ \infty & \text{otherwise.} \end{cases} \quad (2.8)$$

Denote by $L_N(t)$, $t \geq 0$, the generator L_N introduced in (2.3) in which the pair (E, λ) is replaced by $(E(t), \lambda(t))$, and by $\{S_t^N : t \geq 0\}$ the semigroup associated to the generators $N^2 L_N(t)$: $(d/dt)S_t^N = L_N(t)S_t^N$. Note that time has been speeded-up diffusively since the generator has been multiplied by N^2 .

Theorem 2.1. *Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures, $\mu_N \in \mathcal{M}_N$, such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N} H_N(\mu_N | \nu_{\gamma(\cdot)}^N) = 0.$$

Then, for every $t > 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{N} H_N(\mu_N S_t^N | \nu_{\rho(t, \cdot)}^N) = 0.$$

Corollary 2.2. *Under the assumptions of Theorem 2.1, for every $t \geq 0$, every continuous function $H : [0, 1] \rightarrow \mathbb{R}$, and every cylinder function $\Psi : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$,*

$$\lim_{N \rightarrow \infty} E_{\mu_N S_t^N} \left[\left| \frac{1}{N} \sum_{k=1}^{N-1} H(k/N) (\tau_k^{N, \lambda(t)} \Psi)(\eta) - \int_0^1 H(x) E_{\nu_{\rho(t, x)}}[\Psi] dx \right| \right] = 0.$$

2.3. Quasi-static transformations. Fix $\nu > 0$, a function λ in $C^1(\mathbb{R}_+)$, and let $\alpha : \mathbb{R}_+ \rightarrow (0, 1)$ be given by

$$\alpha(t) = f'(\lambda(t)), \quad (2.9)$$

Fix a function $v_0 = v_0^\nu$ in $C_0^2([0, 1])$, and assume that there exists a function ψ in $C^{1+\beta/2, 2+\beta}(\mathbb{R}_+ \times [0, 1])$, for some $\beta > 0$, such that $\psi(t, 0) = \psi(t, 1) = \alpha(t)$, $t \geq 0$, $\psi(0, x) = \alpha(0) + \nu^{-1}v_0(x)$, $x \in [0, 1]$, and

$$\partial_t \psi = \nu \partial_x (D(\psi) \partial_x \psi) \quad \text{at } (t, x) = (0, 0) \text{ and } (t, x) = (0, 1).$$

This means that we assume that

$$\alpha'(0) = \partial_x \left\{ D(\alpha(0) + \nu^{-1}v_0(a)) \partial_x v_0(a) \right\} \quad \text{for } a = 0, 1.$$

Denote by $\rho(t, x) = \rho_\nu(t, x)$ the unique classical solution of the initial-boundary value problem

$$\begin{cases} \partial_t \rho = \nu \partial_x (D(\rho) \partial_x \rho), \\ \rho(t, 0) = \rho(t, 1) = \alpha(t), \\ \rho(0, x) = \alpha(0) + \nu^{-1}v_0. \end{cases}$$

Let $u_\nu : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ be the function given by

$$u_\nu(t, x) = \nu \{ \rho_\nu(t, x) - \alpha(t) \},$$

and, for each $t \geq 0$, let $v_t : [0, 1] \rightarrow \mathbb{R}$ be the unique solution of the linear elliptic equation

$$\begin{cases} \partial_x(D(\alpha(t))\partial_x v_t) = \alpha'(t) , \\ v_t(0) = v_t(1) = 0 . \end{cases} \quad (2.10)$$

Proposition 2.3. *Assume that λ belongs to $C^2(\mathbb{R}_+)$ and that v_0 belongs to $C_0^4([0, 1])$. Then, for each $t \geq 0$,*

$$\lim_{\nu \rightarrow \infty} \int_0^1 [u_\nu(t, x) - v_t(x)]^2 dx = 0 .$$

One can strengthen the topology in which the convergence occurs, but we do not seek optimal conditions here.

Inspired by the previous result, consider a function λ in $C^1(\mathbb{R}_+)$, and let $\alpha : \mathbb{R}_+ \rightarrow (0, 1)$ be given by (2.9). Fix a sequence ϵ_N which vanishes as $N \rightarrow \infty$ and a function $\gamma = \gamma_N$ in $C_0^2([0, 1])$. Assume that there exists a function ψ in $C^{1+\beta/2, 2+\beta}(\mathbb{R}_+ \times [0, 1])$, for some $\beta > 0$, such that $\psi(t, 0) = \psi(t, 1) = \alpha(t)$, $t \geq 0$, $\psi(0, x) = \alpha(0) + \epsilon_N \gamma(x)$, $x \in [0, 1]$, such that

$$\alpha'(a) = \partial_x \left\{ D(\alpha(a) + \epsilon_N \gamma(a)) \partial_x \gamma(a) \right\} \quad \text{for } a = 0, 1 .$$

Denote by $\rho_N(t, x)$ the solution of

$$\begin{cases} \partial_t \rho = \epsilon_N^{-1} \partial_x(D(\rho)\partial_x \rho) , \\ \rho(t, 0) = \rho(t, 1) = \alpha(t) , \\ \rho(0, x) = \alpha(0) + \epsilon_N \gamma(x) . \end{cases} \quad (2.11)$$

Denote by $\mathcal{L}_N(t)$ the generator L_N introduced in (2.3) with $E = 0$ and $\lambda_0 = \lambda_1 = \lambda(t)$. Let $\{T_t^N : t \geq 0\}$ be the semigroup associated to the generator $\epsilon_N^{-1} N^2 \mathcal{L}_N(t)$. Note that time has been speed-up by $\epsilon_N^{-1} N^2$.

Theorem 2.4. *Assume that $\epsilon_N^4 N \rightarrow \infty$, that λ belongs to $C^2(\mathbb{R}_+)$, and that γ belongs to $C_0^4([0, 1])$. Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures, $\mu_N \in \mathcal{M}_N$, such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N \epsilon_N^2} H_N(\mu_N | \nu_{\rho_N(0, \cdot)}^N) = 0 . \quad (2.12)$$

Then, for every $t > 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{N \epsilon_N^2} H_N(\mu_N T_t^N | \nu_{\rho_N(t, \cdot)}^N) = 0 .$$

Corollary 2.5. *Under the assumptions of Theorem 2.4, for every $t \geq 0$, every continuous function $H : [0, 1] \rightarrow \mathbb{R}$, and every cylinder function $\Psi : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$,*

$$\lim_{N \rightarrow \infty} E_{\mu_N T_t^N} \left[\frac{1}{\epsilon_N} \left| \frac{1}{N} \sum_{k=1}^{N-1} H(k/N) (\tau_k^{N, \lambda(t)} \Psi)(\eta) - \int_0^1 H(x) E_{\nu_{v_N(t, x)}}[\Psi] dx \right| \right] = 0 ,$$

where $v_N(t, x) = \alpha(t) + \epsilon_N v(t, x)$, $v(t, x)$ being the unique classical solution of the elliptic equation (2.10).

3. PROOF OF THE MAIN RESULTS

We present in Theorem 3.1 below a general statement from which one can easily deduce Theorems 2.1 and 2.4. For a fixed chemical potential $\lambda = (\lambda_0, \lambda_1)$ and a continuous external field $E : [0, 1] \rightarrow \mathbb{R}$, denote by $\bar{\rho}_{\lambda, E} : [0, 1] \rightarrow \mathbb{R}$ the solution of the elliptic equation

$$\begin{cases} \partial_x(D(\rho)\partial_x\rho) - \partial_x(\chi(\rho)E) = 0, \\ f'(\rho(0)) = \lambda_0, \quad f'(\rho(1)) = \lambda_1, \end{cases} \quad (3.1)$$

Fix sequences $\{\epsilon_N : N \geq 1\}$, $\{\ell_N : N \geq 1\}$ such that $\ell_N \rightarrow \infty$, $\epsilon_N \rightarrow 0$. Consider a time-dependent external field E in $C^{1,2}(\mathbb{R}_+ \times [0, 1])$ and a time-dependent chemical potential $\lambda(t) = (\lambda_0(t), \lambda_1(t))$ such that $\lambda_0, \lambda_1 \in C^1(\mathbb{R}_+)$. Fix a density profile $\gamma = \gamma_N$ in $C^2([0, 1])$ and assume that there exists a function ψ in $C^{1+\beta/2, 2+\beta}(\mathbb{R}_+ \times [0, 1])$, $\beta > 0$, such that $f'(\psi(t, a)) = \lambda_a(t)$ for $a = 0, 1$, $\psi(0, x) = \bar{\rho}_{\lambda(0), E(0)}(x) + \epsilon_N \gamma(x)$ for $x \in [0, 1]$, and such that

$$\partial_t \psi = \ell_N \left\{ \partial_x(D(\psi)\partial_x\psi) - \partial_x(\chi(\psi)E(t)) \right\} \quad \text{at } (t, x) = (0, 0) \text{ and } (t, x) = (0, 1).$$

Denote by $\rho_N(t, \cdot)$ the unique weak solution of the parabolic equation

$$\begin{cases} \partial_t \rho = \ell_N \left\{ \partial_x(D(\rho)\partial_x\rho) - \partial_x(\chi(\rho)E) \right\}, \\ f'(\rho(t, 0)) = \lambda_0(t), \quad f'(\rho(t, 1)) = \lambda_1(t), \\ \rho(0, x) = \bar{\rho}_{\lambda(0), E(0)} + \epsilon_N \gamma(x). \end{cases} \quad (3.2)$$

In Theorem 3.1 the following conditions on the solution of equation (3.2) are needed: For every $T > 0$, there exists $0 < \delta < 1$ such that

$$\delta \leq \rho_N(t, x) \leq 1 - \delta \quad \text{for all } 0 \leq x \leq 1, \quad 0 \leq t \leq T, \quad N \geq 1. \quad (3.3)$$

To explain the second condition, observe that we may rewrite the PDE (3.2) as

$$\partial_t \rho = \ell_N \partial_x \{ \chi(\rho) [\partial_x f'(\rho) - E] \}$$

because $\chi(\rho)f''(\rho) = D(\rho)$ by Einstein relation (2.6). Let

$$F_N(t, x) = \partial_x f'(\rho_N(t, x)) - E(t, x)$$

We assume that for every $T > 0$, there exists a finite constant C_0 such that for all $N \geq 1$, $0 \leq t \leq T$,

$$\|F_N(t)\|_\infty \leq \frac{C_0}{\ell_N}, \quad \|\partial_x F_N(t)\|_\infty \leq \frac{C_0}{\ell_N}. \quad (3.4)$$

Note that for this condition to be fulfilled at $t = 0$, we need $\ell_N \epsilon_N$ to be bounded:

$$\ell_N \epsilon_N \leq C_0 \quad (3.5)$$

for some finite constant C_0 .

Consider two non-decreasing sequences K_N, J_N . We write

$$K_N \ll J_N \text{ if } K_N/J_N \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Recall that we denote by $L_N(t)$ the generator L_N introduced in (2.3) with $E(t)$, $\lambda(t)$ in place of E , λ , respectively. Let $\{\mathfrak{S}_t^N : t \geq 0\}$ be the semigroup associated to the generators $\{\ell_N N^2 L_N(s) : s \geq 0\}$: $(d/dt)\mathfrak{S}_t^N = \ell_N N^2 \mathfrak{S}_t^N L_N(t)$.

Theorem 3.1. *Consider a continuous external field $E(t, x)$ and a continuous chemical potential $\lambda(t) = (\lambda_0(t), \lambda_1(t))$. Assume that γ belongs to $C_0^2([0, 1])$, that conditions (3.3), (3.4), (3.5) hold, and that $\epsilon_N^{-4} \ll N$. Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures, $\mu_N \in \mathcal{M}_N$, such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N \epsilon_N^2} H_N(\mu_N | \nu_{\rho_N(0, \cdot)}^N) = 0. \quad (3.6)$$

Then, for every $t > 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{N \epsilon_N^2} H_N(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t, \cdot)}^N) = 0.$$

The proof of Theorem 3.1 is divided in several steps. Fix a density $\theta \in (0, 1)$, and denote by $\nu_\theta = \nu_\theta^N$ the product measure on Σ_N with density θ :

$$\nu_\theta(\eta) = \frac{1}{Z_N(\theta)} \exp \left\{ f'(\theta) \sum_{j=1}^{N-1} \eta_j \right\}, \quad (3.7)$$

where $Z_N(\theta)$ is the partition function which turns ν_θ into a probability measure, and

$$\beta := f'(\theta) = \log \frac{\theta}{1 - \theta}. \quad (3.8)$$

We use the same notation ν_θ to represent the the Bernoulli product measure on $\{0, 1\}^{\mathbb{Z}}$ with density θ .

Let $L^2(\nu_\theta)$ be the space of functions $f : \Sigma_N \rightarrow \mathbb{R}$ endowed with the scalar product

$$\langle f, g \rangle_{\nu_\theta} = \int f(\eta) g(\eta) \nu_\theta(d\eta).$$

Denote by $L_N^* = L_N^{\lambda, E, *}$ the adjoint in $L^2(\nu_\theta)$ of the generator L_N introduced in (2.3). A simple computation shows that for all $f : \Sigma_N \rightarrow \mathbb{R}$,

$$(L_N^* f)(\eta) = (L_{0,1}^* f)(\eta) + \sum_{j=1}^{N-2} (L_{j,j+1}^* f)(\eta) + (L_{N-1,N}^* f)(\eta), \quad (3.9)$$

where, for $1 \leq j \leq N-2$,

$$\begin{aligned} (L_{N-1,N}^* f)(\eta) &= c_{N-1,N}^{N,\lambda,E}(\sigma^{N-1,N} \eta) e^{\beta(1-2\eta_{N-1})} f(\sigma^{N-1,N} \eta) - c_{N-1,N}^{N,\lambda,E}(\eta) f(\eta), \\ (L_{j,j+1}^* f)(\eta) &= c_{j,j+1}^{N,\lambda,E}(\sigma^{j,j+1} \eta) f(\sigma^{j,j+1} \eta) - c_{j,j+1}^{N,\lambda,E}(\eta) f(\eta), \\ (L_{0,1}^* f)(\eta) &= c_{0,1}^{N,\lambda,E}(\sigma^{0,1} \eta) e^{\beta(1-2\eta_1)} f(\sigma^{0,1} \eta) - c_{0,1}^{N,\lambda,E}(\eta) f(\eta). \end{aligned}$$

In this formula, β is the chemical potential associated to the density θ , which has been introduced in (3.8). It follows from the previous formula that the adjoint of $L_N(t)$ in $L^2(\nu_\theta)$, denoted by $L_N^*(t)$, is given by (3.9) with E and λ replaced by $E(t)$ and $\lambda(t)$.

Proof of Theorem 3.1. Fix sequences ℓ_N, ϵ_N satisfying the assumptions of the theorem, and let γ be a function in $C_0^2([0, 1])$. Denote by $\rho(t, x) = \rho_N(t, x)$ the solution of (3.2). Consider a sequence of probability measures $\{\mu_N : N \geq 1\}$, $\mu_N \in \mathcal{M}_N$, satisfying (3.6). Let $\alpha(t) = (\alpha_0(t), \alpha_1(t))$ be the density of particles associated to the chemical potential $\lambda(t)$:

$$\alpha_0(t) = \frac{e^{\lambda_0(t)}}{1 + e^{\lambda_0(t)}}, \quad \alpha_1(t) = \frac{e^{\lambda_1(t)}}{1 + e^{\lambda_1(t)}}. \quad (3.10)$$

Recall that $\{\mathfrak{S}_t^N : t \geq 0\}$ represents the semigroup associated to the generator $N^2 \ell_N L_N(t)$, and let

$$f_t = \frac{d\mu_N \mathfrak{S}_t^N}{d\nu_\theta}, \quad \psi_t = \frac{d\nu_{\rho_N(t, \cdot)}^N}{d\nu_\theta}. \quad (3.11)$$

A simple computation yields

$$\psi_t(\eta) = \frac{Z_N(\theta)}{Z_N(\rho(t))} \exp \left\{ \sum_{j=1}^{N-1} \eta_j [f'(\rho(t, j/N)) - f'(\theta)] \right\}, \quad (3.12)$$

where $\rho(t, x) = \rho_N(t, x)$ is the solution of equation (3.2), $Z_N(\theta)$, f have been introduced in (3.7), (2.6), respectively, and $Z_N(\rho(t))$ is the normalizing constant given by

$$Z_N(\rho(t)) = \exp \left\{ - \sum_{j=1}^{N-1} \log[1 - \rho(t, j/N)] \right\}.$$

With this notation, in view of (2.8),

$$H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t, \cdot)}) = \int f_t \log \frac{f_t}{\psi_t} d\nu_\theta.$$

Moreover, an elementary computation shows that the density f_t solves the Kolmogorov forward equation

$$\frac{d}{dt} f_t = N^2 \ell_N L_N^*(t) f_t. \quad (3.13)$$

The proof of Theorem 3.1 is divided in three steps.

Step 1: Entropy production. A computation, similar to the one presented in the proof of Lemma 1.4 in [8, Chapter 6], yields that

$$\frac{d}{dt} H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t, \cdot)}) \leq \int \frac{N^2 \ell_N L_N^*(t) \psi_t - \partial_t \psi_t}{\psi_t} f_t d\nu_\theta. \quad (3.14)$$

Let h and $g : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be the cylinder functions given by

$$h(\xi) = \sum_{a=1}^m m_a h_a(\xi), \quad g(\xi) = \frac{1}{2} [\xi_1 - \xi_0]^2 c(\xi), \quad (3.15)$$

where $m_a = \sum_k k \mu_a(k)$. Recall the definition of the operators $\tau_j^{N, \lambda}$, $N \geq 1$, $1 \leq j < N-1$, introduced just above (2.2), and let $\tau_j^N(t) = \tau_j^{N, \lambda(t)}$. A long, but straightforward, computation which uses the identity (2.2), yields that

$$\frac{N^2 \ell_N L_N^*(t) \psi_t - \partial_t \psi_t}{\psi_t} = \ell_N \{I_1 + I_2 + I_3\} + O_N(\ell_N),$$

where $O_N(\ell_N)$ represents an error absolutely bounded by $C_0 \ell_N$, C_0 being a finite constant independent of N , and where

$$\begin{aligned} I_1 &= \sum_{j=1}^{N-2} G_1(t, j/N) (\tau_j^N(t) h)(\eta) + \sum_{j=1}^{N-2} G_2(t, j/N) (\tau_j^N(t) g)(\eta) \\ &\quad - \sum_{j=1}^{N-1} \frac{(\partial_t \rho)}{\chi(\rho)}(t, j/N) [\eta(j) - \rho(t, j/N)], \end{aligned}$$

$$\begin{aligned}
I_2 = & N H_-(t) \sum_{a=1}^m \sum_{k \in \mathbb{Z}} \mu_a(k) \sum_{j=1-k}^0 (\tau_j^N(t) h_a)(\eta) \\
& - N H_+(t) \sum_{a=1}^m \sum_{k \in \mathbb{Z}} \mu_a(k) \sum_{j=N-1-k}^{N-2} (\tau_j^N(t) h_a)(\eta) ,
\end{aligned}$$

$$I_3 = N H_+(t) (\tau_{N-1}^N(t) c)(\eta) [\eta_{N-1} - \alpha_1(t)] - N H_-(t) (\tau_0^N(t) c)(\eta) [\eta_1 - \alpha_0(t)] .$$

In these formulas,

$$\begin{aligned}
G_1(t, x) &= \partial_x \{ \partial_x f'(\rho(t, x)) - E(t, x) \} , \\
G_2(t, x) &= \partial_x f'(\rho(t, x)) \{ \partial_x f'(\rho(t, x)) - E(t, x) \} , \\
H_-(t) &= \partial_x f'(\rho(t, 0)) - E(t, 0) , \quad H_+(t) = \partial_x f'(\rho(t, 1)) - E(t, 1) ,
\end{aligned}$$

and $\alpha_0(t)$, $\alpha_1(t)$ are the densities at the boundary, defined in (3.10). Note that $G_1 = \partial_x F_N$, $G_2 = F_N^2 + E F_N$, $H_-(t) = F_N(t, 0)$ and $H_+(t) = F_N(t, 1)$. In particular, by (3.4), there exists a finite constant C_0 such that for all $N \geq 1$, $0 \leq t \leq T$,

$$\|G_1(t)\|_\infty \leq \frac{C_0}{\ell_N} , \quad \|G_2(t)\|_\infty \leq \frac{C_0}{\ell_N} , \quad \|H_\pm(t)\|_\infty \leq \frac{C_0}{\ell_N} . \quad (3.16)$$

For a cylinder function $\Psi : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$, let $\hat{\Psi} : [0, 1] \rightarrow \mathbb{R}$ be the polynomial given by

$$\hat{\Psi}(\theta) = E_{\nu_\theta}[\Psi] , \quad (3.17)$$

where, we recall, ν_θ is the Bernoulli product measure with density θ . By (3.15), (2.5) and (6.2),

$$\hat{h}'(\theta) = D(\theta) , \quad \hat{g}(\theta) = \chi(\theta) . \quad (3.18)$$

We claim that, in the first two line of I_1 , the replacement of the cylinder functions $\tau_j^N(t)h$, $\tau_j^N(t)g$ by $\tau_j^N(t)h - \hat{h}(\rho(t, j/N))$, $\tau_j^N(t)g - \hat{g}(\rho(t, j/N))$, respectively, produces an error absolutely bounded by a finite constant independent of N . Similarly, the replacement in the two line of I_2 of the cylinder functions $\tau_j^N(t)h_a$, $j \sim 0$, $\tau_k^N(t)h_a$, $k \sim N$, by $\tau_j^N(t)h_a - \hat{h}_a(\alpha_0(t))$, $\tau_k^N(t)h_a - \hat{h}_a(\alpha_1(t))$ produces an error of the same order.

Indeed, denote by J_1 (resp. J_2) the first line of I_1 (resp. the two lines of I_2) with the cylinder functions $\tau_j^N(t)h$, $\tau_j^N(t)g$ (resp. $\tau_j^N(t)h_a$, $j \sim 0$, $\tau_k^N(t)h_a$, $k \sim N$) replaced by $\hat{h}(\rho(t, j/N))$, $\hat{g}(\rho(t, j/N))$ (resp. $\hat{h}_a(\alpha_0(t))$, $\hat{h}_a(\alpha_1(t))$). In the expression of J_2 , observe that $\sum_k k \mu_a(k) = m_a$. For any Lipschitz-continuous function $G : [0, 1] \rightarrow \mathbb{R}$, and for any non-negative integers p, q ,

$$\sum_{j=p}^{N-q} G(j/N) = N \int_0^1 G(x) dx + O_N(1) ,$$

where $O_N(1)$ represents an error absolutely bounded by a finite constant independent of N . It follows from this estimate, from an integration by parts, and from the identities (2.6), (3.18) that $J_1 + J_2$ is absolutely bounded by a finite constant independent of N , proving the claim.

An elementary computation gives that

$$\frac{(\partial_t \rho)}{\chi(\rho)} = \{ \partial_x^2 f'(\rho) - \partial_x E \} D(\rho) + \{ [\partial_x f'(\rho)]^2 - E \partial_x f'(\rho) \} \chi'(\rho) .$$

In conclusion, in view of (3.18), up to this point, we have shown that

$$\frac{N^2 \ell_N L_N^*(t) \psi_t - \partial_t \psi_t}{\psi_t} = \ell_N \{ \hat{I}_1 + \hat{I}_2 + I_3 \} + O(\ell_N), \quad (3.19)$$

where

$$\begin{aligned} \hat{I}_1(t, \eta) &= \sum_{j=1}^{N-2} G_1(t, j/N) V_N(h; t, j, \eta) + \sum_{j=1}^{N-2} G_2(t, j/N) V_N(g; t, j, \eta), \\ \hat{I}_2(t, \eta) &= N H_-(t) \sum_{a=1}^m \sum_{k \in \mathbb{Z}} \mu_a(k) \sum_{j=1-k}^0 \{ (\tau_j^N(t) h_a)(\eta) - \hat{h}_a(\alpha_0(t)) \} \\ &\quad - N H_+(t) \sum_{a=1}^m \sum_{k \in \mathbb{Z}} \mu_a(k) \sum_{j=N-1-k}^{N-2} \{ (\tau_j^N(t) h_a)(\eta) - \hat{h}_a(\alpha_1(t)) \}, \end{aligned}$$

and, for a cylinder function $\varphi : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$,

$$V_N(\varphi; t, j, \eta) = (\tau_j^N(t) \varphi)(\eta) - \hat{\varphi}(\rho(t, j/N)) - \hat{\varphi}'(\rho(t, j/N)) [\eta_j - \rho(t, j/N)].$$

Step 2: A mesoscopic entropy estimate. Denote by $D(\mathbb{R}_+, \Sigma_N)$ the space of right-continuous trajectories $x : \mathbb{R}_+ \rightarrow \Sigma_N$ with left limits. For each probability measure μ in \mathcal{M}_N , denote by \mathbb{P}_μ^N the probability measure on $D(\mathbb{R}_+, \Sigma_N)$ induced by the Markov chain with generator $\ell_N N^2 L_N(t)$ starting from the distribution μ . Expectation with respect to \mathbb{P}_μ^N is represented by \mathbb{E}_μ^N .

Recall that $\epsilon_N^{-4} \ll N$. Let $M_N = \epsilon_N^{-2}$, and fix a sequence $\{K_N : N \geq 1\}$ such that $M_N \ll K_N$, $M_N K_N \ll N$. Let

$$\tilde{I}_{1,N}(t, \eta) = \sum_{j=K_N+1}^{N-K_N-1} G_1(t, j/N) V_N(\hat{h}; t, j, \eta) + \sum_{j=1}^{N-1} G_2(t, j/N) V_N(\hat{g}; t, j, \eta),$$

where, for a smooth function $\hat{\varphi} : [0, 1] \rightarrow \mathbb{R}$,

$$V_N(\hat{\varphi}; t, j, \eta) = \hat{\varphi}(\eta^{K_N}(j)) - \hat{\varphi}(\rho(t, j/N)) - \hat{\varphi}'(\rho(t, j/N)) [\eta^{K_N}(j) - \rho(t, j/N)].$$

Note that in the definition of $\hat{I}_1(t, \eta)$ the sum is carried over $1 \leq j \leq N-1$, while in the definition of $\tilde{I}_1(t, \eta)$ it is carried over $K_N+1 \leq j \leq N-K_N-1$. In view of (3.16), this produces an error of order K_N/ℓ_N in the difference between $\hat{I}_1(t, \eta)$ and $\tilde{I}_1(t, \eta)$.

By (3.16) and Lemma 4.2, since $M_N K_N \ll N$,

$$\lim_{N \rightarrow \infty} \frac{M_N \ell_N}{N} \mathbb{E}_{\mu_N} \left[\int_0^t \{ \hat{I}_{1,N}(s, \eta_s) - \tilde{I}_{1,N}(s, \eta_s) \} ds \right] = 0,$$

and

$$\lim_{N \rightarrow \infty} \frac{M_N \ell_N}{N} \mathbb{E}_{\mu_N} \left[\int_0^t \hat{I}_2(s, \eta_s) ds \right] = 0, \quad \lim_{N \rightarrow \infty} \frac{M_N \ell_N}{N} \mathbb{E}_{\mu_N} \left[\int_0^t I_3(s, \eta_s) ds \right] = 0.$$

On the other hand, by definition of M_N , by (3.5) and by the assumption on ϵ_N , $M_N \ell_N = \epsilon_N^{-2} \ell_N \leq C_0 \epsilon_N^{-3} \ll N$. Therefore, in view of (3.14), (3.19),

$$\begin{aligned} \frac{1}{N \epsilon_N^2} H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t, \cdot)}) &\leq \frac{1}{N \epsilon_N^2} H(\mu_N | \nu_{\rho_N(0, \cdot)}) \\ &\quad + \frac{M_N \ell_N}{N} \mathbb{E}_{\mu_N} \left[\int_0^t \tilde{I}_{1,N}(s, \eta_s) ds \right] + R_N, \end{aligned}$$

where R_N vanishes as $N \rightarrow \infty$.

Step 3: A large deviations estimate. A Taylor expansion up to the second order shows that $V_N(\hat{\varphi}; t, j, \eta)$ is absolutely bounded by $C_0[\eta^{K_N}(j) - \rho(t, j/N)]^2$. The second term on the right hand side of the previous equation is thus bounded above by

$$C_0 \mathbb{E}_{\mu_N} \left[\int_0^t \frac{M_N \ell_N}{N} \sum_{j=K_N+1}^{N-K_N-1} J(s, j/N) [\eta_s^{K_N}(j) - \rho(s, j/N)]^2 ds \right],$$

where $J(t, x) = |G_1(t, x)| + |G_2(t, x)|$. Since $M_N = \epsilon_N^{-2}$, by the entropy inequality, the previous expression is less than or equal to

$$\begin{aligned} & \frac{C_0}{AN\epsilon_N^2} \int_0^t H(\mu_N \mathfrak{S}_s^N | \nu_{\rho_N(s, \cdot)}) ds \\ & + \int_0^t \frac{\epsilon_N^{-2}}{AN} \log E_{\nu_{\rho_N(s, \cdot)}} \left[\exp \left\{ A \ell_N \sum_{j=K_N+1}^{N-K_N-1} J(s, j/N) [\eta^{K_N}(j) - \rho(s, j/N)]^2 \right\} \right] ds \end{aligned}$$

for every $A > 0$. By Hölder's inequality and since $\nu_{\rho_N(s, \cdot)}$ is a product measure, the second term of the last sum is less than or equal to

$$\int_0^t \frac{\epsilon_N^{-2}}{ANK_N} \sum_{j=K_N+1}^{N-K_N-1} \log E_{\nu_{\rho_N(s, \cdot)}} \left[\exp \left\{ A \ell_N J(s, j/N) K_N [\eta^{K_N}(j) - \rho(s, j/N)]^2 \right\} \right] ds$$

By (3.3) and (3.16), $\ell_N J(s, j/N) \leq C_0$ and $\delta \leq \rho_N(s, x) \leq 1 - \delta$ for some $\delta > 0$. Therefore, since $\nu_{\rho_N(s, \cdot)}$ is the product measure with density $\rho_N(s, \cdot)$, there exists A_0 such that for

$$E_{\nu_{\rho_N(s, \cdot)}} \left[\exp \left\{ AC_0 K_N [\eta^{K_N}(j) - \rho(s, j/N)]^2 \right\} \right] \leq C'_0$$

for all $0 < A \leq A_0$. The previous integral is therefore less than or equal to $C_0 \epsilon_N^{-2} / AK_N \ll 1$. This proves that there exists a finite constant C_0 such that

$$\begin{aligned} \frac{1}{N\epsilon_N^2} H(\mu_N \mathfrak{S}_t^N | \nu_{\rho_N(t, \cdot)}) & \leq \frac{1}{N\epsilon_N^2} H(\mu_N | \nu_{\rho_N(0, \cdot)}) \\ & + C_0 \int_0^t \frac{1}{N\epsilon_N^2} H(\mu_N \mathfrak{S}_s^N | \nu_{\rho_N(s, \cdot)}) ds + R_N, \end{aligned}$$

where R_N vanishes as $N \rightarrow \infty$. To conclude the proof of Theorem 3.1 it remains to apply Gronwall inequality. \square

Proof of Theorem 2.1. Set $\epsilon_N = \ell_N = 1$. (3.3). Conditions (3.4) and (3.5) are trivially satisfied. The assertion of Theorem 2.1 follows therefore from Theorem 3.1. \square

Proof of Theorem 2.4. Assume that the external field vanishes: $E(t, x) = 0$ and that the left and right chemical potentials are equal, $\lambda_0(t) = \lambda_1(t)$, $t \geq 0$. In This case the solution of the elliptic equation (3.1) $\bar{\rho}_{\lambda,0}$ is constant in space, $\bar{\rho}_{\lambda,0}(x) = \alpha$, where $\alpha = f'(\lambda)$.

Condition (3.3) for N large enough follows from Proposition 5.1. Condition (3.4), which can be read as conditions on $\partial_x \rho_N$ and $\partial_x^2 \rho_N$, follows from Propositions 5.1 and 5.2. \square

Proofs of Corollary 2.2 and 2.5. The proofs are analogous to the one of Corollary 1.3, Chapter 6 in [8], provided we replace in the statement of Corollary 2.5 $\nu_{v_N(t,x)}$ by $\nu_{\rho_N(t,x)}$. However, since Ψ is a cylinder function,

$$\frac{1}{\epsilon_N} \left| \int_0^1 H(x) \{ E_{\nu_{v_N(t,x)}}[\Psi] - E_{\nu_{\rho_N(t,x)}}[\Psi] \} dx \right| \leq \frac{C_0}{\epsilon_N} \int_0^1 |v_N(t,x) - \rho_N(t,x)| dx .$$

By definition of v_N , the right hand side is equal to

$$C_0 \int_0^1 |u_N(t,x) - v(t,x)| dx ,$$

where $u_N(t) = \epsilon_N^{-1}[\rho_N(t) - \alpha(t)]$. It remains to recall the statement of Proposition 2.3. \square

4. ENTROPY ESTIMATES

We adopt in this section the notation and the set-up introduced in the previous one. Recall from (3.7) that $\theta \in (0, 1)$ is a fixed parameter and that ν_θ is the product measure on Σ_N with density θ . It is not difficult to show that there exists a finite constant $C_0 = C_0(\theta)$ such that

$$\sup_{\mu} H(\mu|\nu_\theta) \leq C_0 N ,$$

where the supremum is carried over all probability measures μ on Σ_N .

Fix a smooth function $\hat{\lambda} : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ such that $\hat{\lambda}(t, a) = \lambda_a(t)$, $t \geq 0$, $a = 0$, 1. Let $\alpha(t, x) = e^{\hat{\lambda}(t,x)} / [1 + e^{\hat{\lambda}(t,x)}]$, and denote by $\nu_{\alpha(t)}^N$, $t \geq 0$, the product measure on Σ_N associated to the density $\alpha(t, x)$:

$$\nu_{\alpha(t)}^N(\eta) = \frac{1}{\hat{Z}_N(t)} \exp \left\{ \sum_{j=1}^{N-1} \eta_j f'(\alpha(t, j/N)) \right\} ,$$

where $\hat{Z}_N(t)$ is the normalizing constant given by

$$\hat{Z}_N(t) = \exp \left\{ - \sum_{j=1}^{N-1} \log[1 - \alpha(t, j/N)] \right\} .$$

Note that $\alpha(t, x)$ takes values in $(0, 1)$. In particular, the quantities introduced above are well defined.

Recall that $\{\mathfrak{S}_t^N : t \geq 0\}$ represents the semigroup associated to the generator $N^2 \ell_N L_N(t)$, $(d/dt)\mathfrak{S}_t^N = L_N(t)\mathfrak{S}_t^N$, and that $H_N(\mu|\pi)$ stands for the relative entropy of μ with respect to π . Denote by $D_N^{\alpha(t)}(\cdot)$, $t \geq 0$, the functional which acts on functions $h : \Sigma_N \rightarrow \mathbb{R}$, as

$$\begin{aligned} D_N^{\alpha(t)}(h) &= \sum_{j=1}^{N-2} \int c_{j,j+1}^{N,\lambda(t)}(\eta) [h(\sigma^{j,j+1}\eta) - h(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) \\ &+ \int c_{0,1}^{N,\lambda(t)}(\eta) r_{0,1}^{\lambda(t)}(\eta) [h(\sigma^{0,1}\eta) - h(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) \\ &+ \int c_{N-1,N}^{N,\lambda(t)}(\eta) r_{N-1,N}^{\lambda(t)}(\eta) [h(\sigma^{N-1,N}\eta) - h(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) , \end{aligned}$$

Lemma 4.1. *Fix a sequence $\{\mu_N : N \geq 1\}$ of probability measures, $\mu_N \in \mathcal{M}_N$. For every $T > 0$, there exists a finite constant C_0 , depending only on $E(t)$, $\alpha(t)$, $0 \leq t \leq T$, such that for all $0 \leq t \leq T$,*

$$H_N(\mu_N \mathfrak{S}_t^N | \nu_{\alpha(t)}^N) \leq -\frac{N^2 \ell_N}{2} \int_0^t D_N^{\alpha(s)}(\sqrt{g_s}) ds + C_0 N \ell_N ,$$

where $g_t = g_t^N = d\mu_N \mathfrak{S}_t^N / d\nu_{\alpha(t)}^N$.

Proof. In this proof, C_0 represents a finite constant which may depend only on θ , $E(t)$, $\alpha(t)$, $0 \leq t \leq T$, but not on N .

Fix a sequence $\{\mu_N : N \geq 1\}$ of probability measures, $\mu_N \in \mathcal{M}_N$. Recall the definition of $f_t = f_t^N$, introduced in (3.11), and let ϕ_t , $t \geq 0$, be given by

$$\phi_t = \frac{d\nu_{\alpha(t)}^N}{d\nu_\theta} \quad \text{so that} \quad g_t = \frac{f_t}{\phi_t} .$$

By definition,

$$H_N(t) := H(\mu_N \mathfrak{S}_t^N | \nu_{\alpha(t)}^N) = \int f_t \log \frac{f_t}{\phi_t} d\nu_\theta ,$$

so that

$$\frac{d}{dt} H_N(t) = N^2 \ell_N \int f_t L_N(t) \log \frac{f_t}{\phi_t} d\nu_\theta - \int f_t \partial_t \log \phi_t d\nu_\theta .$$

The second term on the right hand side is clearly bounded by $C_0 N$. On the other hand, since $a \log b/a \leq 2\sqrt{a}(\sqrt{b} - \sqrt{a})$, for all $a, b > 0$, the first term on the right hand side is less than or equal to

$$2N^2 \ell_N \int h_t L_N(t) h_t d\nu_{\alpha(t)}^N ,$$

where $h_t = \sqrt{g_t} = \sqrt{f_t/\phi_t}$.

Recall the definition of the generator $L_N(t)$ introduced in (2.3). Denote by $L_N^o(t)$ the piece of $L_N(t)$ which corresponds to the sum for j in the range $1 \leq j \leq N-2$, and denote by $L_N^b(t)$ the remaining two terms. A change of variables $\eta' = \sigma^{j,j+1}\eta$, $1 \leq j \leq N-2$, yields

$$2 \int h_t L_N^o(t) h_t d\nu_{\alpha(t)}^N = - \sum_{j=1}^{N-2} \int c_{j,j+1}^{N,\lambda(t)}(\eta) [h_t(\sigma^{j,j+1}\eta) - h_t(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) + R_N ,$$

where R_N is a remainder absolutely bounded by

$$\frac{C_0}{N} \sum_{j=1}^{N-2} \int c_{j,j+1}^{N,\lambda(t)}(\eta) [h_t(\sigma^{j,j+1}\eta) + h_t(\eta)] |h_t(\sigma^{j,j+1}\eta) - h_t(\eta)| d\nu_{\alpha(t)}^N(\eta)$$

for some finite constant C_0 . By Young's inequality, and since $g_t = h_t^2$ is a density with respect to $\nu_{\alpha(t)}^N$, the previous expression is bounded by the sum of a term which can be absorbed by the first term on the right hand side of the penultimate displayed equation with a term bounded by C_0/N , that is,

$$2 \int h_t L_N^o(t) h_t d\nu_{\alpha(t)}^N \leq -\frac{1}{2} \sum_{j=1}^{N-2} \int c_{j,j+1}^{N,\lambda(t)}(\eta) [h_t(\sigma^{j,j+1}\eta) - h_t(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) + \frac{C_0}{N} .$$

Since $\tilde{\lambda}(t)$ is equal to $\lambda(t)$ at the boundary of the interval $[0, 1]$

$$\frac{\nu_{\alpha(t)}^N(\sigma^{0,1}\eta)}{\nu_{\alpha(t)}^N(\eta)} \frac{r_{0,1}^{\lambda(t)}(\sigma^{0,1}\eta)}{r_{0,1}^{\lambda(t)}(\eta)} = 1 + R_N ,$$

where R_N is absolutely bounded by C_0/N . In view of this identity, and with a similar computation to the one presented for the interior piece of the generator, we conclude that

$$\begin{aligned} 2 \int h_t L_N^b(t) h_t d\nu_{\alpha(t)}^N &\leq -\frac{1}{2} \int c_{0,1}^{N,\lambda(t)}(\eta) r_{0,1}^{\lambda(t)}(\eta) [h_t(\sigma^{0,1}\eta) - h_t(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) \\ &\quad - \frac{1}{2} \int c_{N-1,N}^{N,\lambda(t)}(\eta) r_{N-1,N}^{\lambda(t)}(\eta) [h_t(\sigma^{N-1,N}\eta) - h_t(\eta)]^2 d\nu_{\alpha(t)}^N(\eta) + \frac{C_0}{N^2} . \end{aligned}$$

It follows from the previous estimates that

$$H_N(t) - H_N(0) \leq -\frac{N^2 \ell_N}{2} \int_0^t D_N(s, \sqrt{g_s}) ds + C_0 N \ell_N ,$$

which concludes the proof of the lemma since $H_N(0) \leq C_0 N$, as observed at the beginning of this section. \square

For a positive integer k , denote by $\eta^k(j)$ the density of particles in an interval of length $2k+1$ centered at j :

$$\eta^k(j) = \frac{1}{2k+1} \sum_{i \in I_k(j) \cap \Lambda_N} \eta(i) ,$$

where $I_k(j) = \{j-k, \dots, j+k\}$.

Recall the definition of the polynomial $\hat{h} : [0, 1] \rightarrow \mathbb{R}$ given in (3.17), where h is a cylinder function, and recall the definition of the probability measures \mathbb{P}_μ^N introduced at the beginning of Step 2 in the previous section.

Lemma 4.2. *Let $G_N : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ (resp. $H_N : \mathbb{R}_+ \rightarrow \mathbb{R}$), $N \geq 1$, be a sequence of functions in $C^{0,1}(\mathbb{R}_+ \times [0, 1])$ (resp. $C(\mathbb{R}_+)$) such that for all $T > 0$,*

$$\sup_{N \geq 1} \sup_{0 \leq t \leq T} \|G_N(t)\|_\infty < \infty , \quad \sup_{N \geq 1} \sup_{0 \leq t \leq T} \|H_N(t)\|_\infty < \infty .$$

Let $h : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$ be a cylinder function. Fix a sequence $\{\mu_N : N \geq 1\}$ of probability measures, $\mu_N \in \mathcal{M}_N$. Consider two sequences $M_N \uparrow \infty$ and $K_N \uparrow \infty$ such that $M_N \ll K_N$, $M_N K_N \ll N$. Then, for every $T > 0$,

$$\lim_{N \rightarrow \infty} M_N \mathbb{E}_{\mu_N} \left[\int_0^T \frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s, j/N) \{(\tau_j h)(\eta_s) - \hat{h}(\eta_s^{K_N}(j))\} ds \right] = 0 ,$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} M_N \mathbb{E}_{\mu_N} \left[\int_0^T H_N(s) \{ \tau_0^{N,\lambda(s)} h(\eta_s) - \hat{h}(\alpha(s, 0)) \} ds \right] &= 0 , \\ \lim_{N \rightarrow \infty} M_N \mathbb{E}_{\mu_N} \left[\int_0^T H_N(s) \{ \tau_N^{N,\lambda(s)} h(\eta_s) - \hat{h}(\alpha(s, 1)) \} ds \right] &= 0 . \end{aligned}$$

Proof. Fix $T > 0$ and $0 \leq t \leq T$. Every cylinder function can be written as a linear combination of the functions $\Psi_A = \prod_{j \in A} \eta_j$, A a finite subset of \mathbb{Z} . It is therefore enough to prove the lemma for such functions. We present the details for $h = \Psi_{\{0,1\}}$, it will be clear that the arguments apply to all cases.

Fix a sequence of continuous function $G_N : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ satisfying the assumptions of the lemma and note that $\hat{h}(\theta) = \theta^2$ in the case where $h = \Psi_{\{0,1\}}$. It follows from the assumptions of the lemma and from a summation by parts that

$$\frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s, j/N) \left\{ \eta(j)\eta(j+1) - \frac{1}{2K_N+1} \sum_{k=-K_N}^{K_N} \eta(j+k)\eta(j+k+1) \right\}$$

in the time interval $[0, T]$ is absolutely bounded by a term of order K_N/N . On the other hand, we may write the difference $(2K_N+1)^{-1} \sum_{|k| \leq K_N} \eta(j+k)\eta(j+k+1) - \hat{h}(\eta^{K_N}(j))$ as

$$\frac{1}{(2K_N+1)^2} \sum_{k,\ell} \eta(j+k) [\eta(j+k+1) - \eta(j+\ell)] + O\left(\frac{1}{K_N}\right),$$

where the sum is carried over all k, ℓ such that $|k| \leq K_N, |\ell| \leq K_N, k \neq \ell$. The error term takes into account the diagonal terms $k = \ell$. Denote by $V_{j,K_N}(\eta)$ the first term of the previous formula.

In view of the former estimates, the first expectation appearing in the statement of the lemma is equal to

$$\mathbb{E}_{\mu_N} \left[\int_0^T \frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s, j/N) V_{j,K_N}(\eta_s) ds \right] + R_N,$$

where R_N is a remainder absolutely bounded by $C_0\{(K_N/N) + (1/K_N)\}$. Here and below, C_0 is a finite constant which does not depend on N , and which may change from line to line.

Recall the definition of the density g_s , introduced at the beginning of the proof of Lemma 4.1. The first term of the previous formula is equal to

$$\int_0^T ds \int \frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s, j/N) V_{j,K_N}(\eta) g_s(\eta) \nu_{\alpha(s)}^N(d\eta). \quad (4.1)$$

Recall the definition of $V_{j,K_N}(\eta)$ and represent the previous integral, denoted by I , as $(1/2)I + (1/2)I$. In one of the halves, perform the change of variables $\eta' = \sigma^{j+k+1,j+\ell}\eta$ to rewrite the previous expression as

$$\begin{aligned} & \frac{1}{2} \frac{1}{(2K_N+1)^2} \sum_{k,\ell} \int_0^T ds \int \frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s, j/N) \eta(j+k) \times \\ & \times [\eta(j+k+1) - \eta(j+\ell)] \{g_s(\eta) - g_s(\sigma^{j+k+1,j+\ell}\eta)\} \nu_{\alpha(s)}^N(d\eta) + R_N. \end{aligned}$$

In this formula, R_N is a remainder which appears from the change of measures $\nu_{\alpha(s)}^N(\sigma^{j+k+1,j+\ell}\eta)/\nu_{\alpha(s)}^N(\eta)$, and which is bounded by $C_0 K_N/N$. Rewrite $g_s(\eta) - g_s(\eta')$ as $[\sqrt{g_s(\eta)} - \sqrt{g_s(\eta')}] [\sqrt{g_s(\eta)} + \sqrt{g_s(\eta')}]$ and apply Young's inequality to estimate the previous expression by

$$\begin{aligned} & \frac{1}{4A} \frac{1}{\tilde{K}_N^2} \sum_{k,\ell} \int_0^T ds \int \frac{1}{N} \sum_j G_N(s, j/N)^2 \left\{ \sqrt{g_s(\eta)} + \sqrt{g_s(\sigma^{j+k,j+\ell}\eta)} \right\}^2 \nu_{\alpha(s)}^N(d\eta) \\ & + \frac{A}{4} \frac{1}{\tilde{K}_N^2} \sum_{k,\ell} \int_0^T ds \int \frac{1}{N} \sum_j \left\{ \sqrt{g_s(\eta)} - \sqrt{g_s(\sigma^{j+k,j+\ell}\eta)} \right\}^2 \nu_{\alpha(s)}^N(d\eta) \end{aligned}$$

for every $A > 0$. In this formula, $\tilde{K}_N = 2K_N + 1$. Since g_s is a density with respect to $\nu_{\alpha(s)}^N$, the first term of the previous expression is bounded by

$$\frac{C_0}{A} \int_0^T ds \frac{1}{N} \sum_{j=K_N+1}^{N-1-K_N} G_N(s, j/N)^2 \leq \frac{C_0}{A}.$$

On the other hand, by the path lemma, explained in pages 94-95 of [8] and in details below equation (3.7) in [9], the second term of the previous formula is bounded above by

$$\frac{C_0 A K_N^2}{N} \int_0^T ds D_N^{\alpha(s)}(\sqrt{g_s}).$$

Recall that in the path lemma, a change of variables $\eta' = \sigma^{j,j+1} \sigma^{j+1,j+2} \dots \sigma^{k-1,k} \eta$ is performed. Usually, the Jacobian of this change of variables is equal to 1 because the reference measure is a homogeneous product measure. In the present case, where the measure $\nu_{\alpha(s)}^N$ is a local equilibrium, the Jacobian is equal to $\exp\{h(\eta)\}$, where h is uniformly bounded by K_N/N . By Lemma 4.1, the previous displayed equation is less than or equal to $C_0 A (K_N/N)^2$. Optimizing over A , we conclude that (4.1) is bounded by $C_0 K_N/N$.

To complete the proof of the first assertion of the lemma it remains to recollect all the previous estimates and to recall the assumptions on the sequences M_N and K_N .

We turn to the second assertion. Here again, we present the proof for the left boundary in the case where $h(\eta) = \eta_1 \eta_2$. Note that, by definition of $\tau_0^{N,\lambda}$, the case $h(\eta) = \eta_0 \eta_1$ reduces to the case $h(\eta) = \eta_1$.

By definition of g_s , the expectation appearing in the statement of the lemma is equal to

$$\int_0^T ds H_N(s) \int \{ \eta_1 \eta_2 - \alpha(s, 0)^2 \} g_s(\eta) \nu_{\alpha(s)}^N(d\eta).$$

Fix s and write the difference $E_{\nu_{\alpha(s)}^N} [\eta_1 \eta_2 g_s] - \alpha(s, 0)^2$ as

$$\begin{aligned} & \left\{ E_{\nu_{\alpha(s)}^N} [\eta_1 \eta_2 g_s] - E_{\nu_{\alpha(s)}^N} [\eta_2 g_s] \alpha(s, 0) \right\} \\ & + \left\{ E_{\nu_{\alpha(s)}^N} [\eta_2 g_s] \alpha(s, 0) - E_{\nu_{\alpha(s)}^N} [g_s] \alpha(s, 0)^2 \right\}. \end{aligned} \quad (4.2)$$

Since $1 = \eta_1 + (1 - \eta_1)$, the first term inside braces can be written as

$$[1 - \alpha(s, 0)] E_{\nu_{\alpha(s)}^N} [\eta_1 \eta_2 g_s] - \alpha(s, 0) E_{\nu_{\alpha(s)}^N} [(1 - \eta_1) \eta_2 g_s].$$

Performing a change of variables $\eta' = \sigma^{0,1} \eta$ in the first expectation, this difference becomes

$$\alpha(s, 1/N) \frac{(1 - \alpha(s, 0))}{1 - \alpha(s, 1/N)} E_{\nu_{\alpha(s)}^N} [(1 - \eta_1) \eta_2 g_s(\sigma^{0,1} \eta)] - \alpha(s, 0) E_{\nu_{\alpha(s)}^N} [(1 - \eta_1) \eta_2 g_s],$$

Since $|\alpha(s, 1/N) - \alpha(s, 0)| \leq C_0/N$, the previous expression is equal to

$$\alpha(s, 0) E_{\nu_{\alpha(s)}^N} [(1 - \eta_1) \eta_2 \{g_s(\sigma^{0,1} \eta) - g_s(\eta)\}] + R_N,$$

where R_N is a remainder bounded by C_0/N in view of the assumptions on the sequence H_N . At this point, we may repeat the arguments presented in the first part of the proof to bound the first term by $C_0 \{D_N^{\alpha(s)}(\sqrt{g_s})\}^{1/2}$, whose time integral,

in view of Lemma 4.1, is bounded by $C_0 N^{-1/2}$. A similar argument permits to estimate the second term in (4.2). This completes the proof of the lemma. \square

5. THE HYDRODYNAMIC EQUATION

We prove in this section Proposition 2.3 and some estimates, stated below in Propositions 5.1 and 5.2, on the solution of equation (5.1). Recall the definition of the spaces $C^k([0, 1])$ and $C_0^k([0, 1])$, $k \geq 1$, introduced just below (2.6). Denote by $\|f\|_p$, $p \geq 1$, the L^p -norm of a function $f : [0, 1] \rightarrow \mathbb{R}$,

$$\|f\|_p^p = \int_0^1 |f(x)|^p dx .$$

Fix $\nu > 0$, a smooth function $\alpha : \mathbb{R}_+ \rightarrow (0, 1)$, and an initial condition ρ_0 in $C^4([0, 1])$ such that $\rho_0(0) = \rho_0(1) = \alpha(0)$. Denote by $\rho(t, x) = \rho_\nu(t, x)$ the solution of the initial-boundary value problem

$$\begin{cases} \partial_t \rho = \nu \partial_x (D(\rho) \partial_x \rho) , \\ \rho(t, 0) = \rho(t, 1) = \alpha(t) , \\ \rho(0, x) = \rho_0 . \end{cases} \quad (5.1)$$

Proposition 5.1. *For every $t_0 \geq 0$, there exists $\nu_0 < \infty$, such that for all $\nu \geq \nu_0$, there exist positive constants $0 < b < B < \infty$, depending only on D , $\alpha(t)$, $0 \leq t \leq t_0$, such that for all $0 \leq t \leq t_0$,*

$$\begin{aligned} \|\rho(t) - \alpha(t)\|_\infty^2 &\leq B e^{-b\nu t} \|\partial_x \rho_0\|_2^2 + \frac{B}{\nu^2} , \\ \|(\partial_x \rho)(t)\|_\infty^2 &\leq B e^{-b\nu t} \left\{ \|\partial_x^2 \rho_0\|_2^2 + \|\partial_x \rho_0\|_4^4 + \frac{1}{\nu} \|\partial_x \rho_0\|_2^2 \right\} + \frac{B}{\nu^2} . \end{aligned}$$

In this proposition, $e^{-b\nu t}$ corresponds to the speed of convergence to equilibrium of the solution of (5.1) in the case where the boundary condition $\alpha(t)$ does not change in time, while $1/\nu^2$ stands for the relaxation time due to the evolution of the boundary conditions.

Proposition 5.2. *Assume that $\rho_0 = \alpha(0) + \varepsilon v_0$, where v_0 belongs to $C_0^4([0, 1])$. For every $t_0 \geq 0$, there exists $\varepsilon_0 > 0$ and $\nu_0 < \infty$, depending on D , v_0 , $\alpha(t)$, $0 \leq t \leq t_0$, such that for all $\varepsilon < \varepsilon_0$, $\nu \geq \nu_0$, there exist positive constants $B < \infty$, such that for all $0 \leq t \leq t_0$,*

$$\|(\partial_x^2 \rho)(t)\|_\infty^2 \leq B \left\{ \varepsilon^2 + \frac{1}{\nu^4} \right\} .$$

The proof of these propositions is divided in a sequence of assertions. The Poincaré's inequality plays a fundamental role in the argument. It states that there exists a finite constant K_1 such that for every $C^1([0, 1])$ function f which vanishes at some point $x \in [0, 1]$,

$$\int_0^1 f(x)^2 dx \leq K_1 \int_0^1 [f'(x)]^2 dx .$$

Throughout this subsection, c_0 , C_0 represent small and large constants which depend only on K_1 and D .

Let

$$\beta_1(t) = \sup_{0 \leq s \leq t} |\alpha'(s)| , \quad \beta_2(t) = \sup_{0 \leq s \leq t} |\alpha''(s)| . \quad (5.2)$$

Assertion 5.A. *There exist positive constants $0 < c_0 < C_0 < \infty$ such that for all $t \geq 0$,*

$$\int_0^1 [\rho(t) - \alpha(t)]^2 dx \leq e^{-c_0 \nu t} \int_0^1 [\rho(0) - \alpha(0)]^2 dx + \frac{C_0}{\nu^2} \beta_1(t)^2 (1 - e^{-c_0 \nu t}) .$$

Proof. The proof follows classical arguments. Since $\rho(t) = \alpha(t)$ at the boundary, an integration by parts and the fact that $\alpha(t)$ is space independent yield that

$$\frac{1}{2} \frac{d}{dt} \int_0^1 [\rho(t) - \alpha(t)]^2 dx = -\nu \int_0^1 D(\rho(t)) (\partial_x \rho(t))^2 dx - \alpha'(t) \int_0^1 [\rho(t) - \alpha(t)] dx .$$

Since the diffusivity is bounded below by a strictly positive constant, in the first term we may replace $D(\rho(t))$ by c_0 and the identity by an inequality. By Poincaré's inequality, the integral of $-(\partial_x \rho(t))^2$ is bounded by the integral of $-K_1^{-1}[\rho(t) - \alpha(t)]^2$. The second term on the right hand side can be estimated by Young's inequality. One of the terms is absorbed by what remained of the first term. The other one is $(C_0/\nu)\alpha'(t)^2$.

Up to this point we have shown that

$$\frac{1}{2} \frac{d}{dt} \int_0^1 [\rho(t) - \alpha(t)]^2 dx \leq -c_0 \nu \int_0^1 [\rho(t) - \alpha(t)]^2 dx + \frac{C_0}{\nu} \alpha'(t)^2 .$$

To complete the proof, it remains to apply Gronwall inequality. \square

Let $d : [0, 1] \rightarrow \mathbb{R}$ be a primitive of D , $d' = D$, and let

$$c_1 = \inf_{0 \leq \alpha \leq 1} D(\alpha) , \quad C_1 = \|(\log D)'\|_\infty . \quad (5.3)$$

Assertion 5.B. *Assume that $2K_1 C_1 \beta_1(t_0) < c_1 \nu$ for some $t_0 > 0$. Then, there exists a positive constants $C_0 < \infty$ such that for all $0 \leq t \leq t_0$,*

$$\int_0^1 [\partial_x d(\rho(t))]^2 dx \leq e^{-a_\nu t} \int_0^1 [\partial_x d(\rho(0))]^2 dx + \frac{C_0}{\nu} \int_0^t e^{-a_\nu(t-s)} \beta_1(s)^2 ds ,$$

where $a_\nu = (c_1/K_1)\nu - 2C_1\beta_1(t_0)$.

Proof. The proof is similar to the previous one. Adding and subtracting $\alpha'(t)$ we have that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 [\partial_x d(\rho(t))]^2 dx &= \int_0^1 \partial_x d(\rho(t)) \partial_x \left\{ D(\rho(t)) [\nu \partial_x^2 d(\rho(t)) - \alpha'(t)] \right\} dx \\ &\quad + \alpha'(t) \int_0^1 \partial_x d(\rho(t)) \partial_x D(\rho(t)) dx . \end{aligned}$$

Since $\alpha(t) = \rho(t, 0) = \rho(t, 1)$, $\alpha'(t) = \nu \partial_x^2 d(\rho(t, 0)) = \nu \partial_x^2 d(\rho(t, 1))$. In particular, we may integrate by parts the first term on the right hand side. This operation yields a negative term and one involving $\alpha'(t)$. This latter expression can be estimated through Young's inequality. The first piece is absorbed into the negative term and the second piece is bounded by $(C_0/\nu)\alpha'(t)^2$. Hence,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 [\partial_x d(\rho(t))]^2 dx &\leq -\frac{c_1 \nu}{2} \int_0^1 [\partial_x^2 d(\rho(t))]^2 dx + \frac{C_0}{\nu} \alpha'(t)^2 \\ &\quad + C_1 |\alpha'(t)| \int_0^1 [\partial_x d(\rho(t))]^2 dx . \end{aligned}$$

Since $\int_0^1 \partial_x d(\rho(t)) dx = 0$, applying Poincaré inequality to the first term on the right hand side, we obtain that the last expression is bounded above by

$$-\left[\frac{c_1 \nu}{2K_1} - C_1 \beta_1(t)\right] \int_0^1 [\partial_x d(\rho(t))]^2 dx + \frac{C_0}{\nu} \beta_1(t)^2.$$

To complete the proof, it remains to replace $\beta_1(t)$ by $\beta_1(t_0)$ in the term inside brackets, getting an expression which is positive by assumption, and to apply Gronwall inequality. \square

Lemma 5.3. *Assume that $c_1 \nu > 2K_1 C_1 \beta_1(t_0)$ for some $t_0 > 0$. Then, there exist positive constants $0 < c_0 < C_0 < \infty$ such that for all $0 \leq t \leq t_0$,*

$$\|\rho(t) - \alpha(t)\|_\infty^2 \leq C_0 e^{-c_0 \nu t} e^{C_0 \beta(t_0) t} \int_0^1 [\partial_x d(\rho(0))]^2 dx + \frac{1}{\nu^2} \frac{C_0 \beta_1(t)^2}{1 - (A_1 \beta_1(t_0)/\nu)},$$

where $A_1 = 2K_1 C_1 / c_1$.

Proof. Assume that $2K_1 C_1 \beta_1(t_0) < c_1 \nu$ for some $t_0 > 0$ and fix $0 < t \leq t_0$. Since $\alpha(t) = \rho(t, 0)$, by Schwarz inequality there exists a finite constant C_0 such that for every $x \in [0, 1]$,

$$|\rho(t, x) - \alpha(t)|^2 \leq C_0 \int_0^1 [\partial_x \rho(t)]^2 dx \leq C_0 \int_0^1 [\partial_x d(\rho(t))]^2 dx.$$

To complete the proof, it remains to recall Assertion 5.B and to estimate the term $\beta_1(s)^2$ appearing in the time integral by $\beta_1(t)^2$. \square

Let $F_n, G_n : \mathbb{R}_+ \rightarrow \mathbb{R}$, $n \geq 1$, be given by

$$F_n(t) = \int_0^1 [\partial_x d(\rho(t))]^{2n} dx, \quad G_n(t) = \int_0^1 [\partial_x^2 d(\rho(t))]^2 [\partial_x d(\rho(t))]^{2n} dx. \quad (5.4)$$

Assertion 5.C. *For all $n \geq 2$, there exist positive constants $0 < c_0 < C_0 < \infty$, $b_0 > 0$, such that for all $0 < b < b_0$, $t \geq 0$,*

$$\begin{aligned} F_n(t) + c_0 n^2 \nu \int_0^t G_{n-1}(s) e^{-b\nu(t-s)} ds \\ \leq e^{-b\nu t} F_n(0) + C_0 \frac{n^2}{\nu} \beta_1(t)^2 \int_0^t F_{n-1}(s) e^{-b\nu(t-s)} ds. \end{aligned}$$

Proof. Since $\alpha'(t) = \partial_t \rho(t, 1) = \nu \partial_x^2 d(\rho(t, 1))$, adding and subtracting $\alpha'(t)$, and then integrating by parts yield that

$$\begin{aligned} F'_n(t) &= -2n(2n-1) \nu \int_0^1 D(\rho(t)) [\partial_x d(\rho(t))]^{2n-2} [\partial_x^2 d(\rho(t))]^2 dx \\ &\quad + 2n(2n-1) \alpha'(t) \int_0^1 D(\rho(t)) [\partial_x d(\rho(t))]^{2n-2} \partial_x^2 d(\rho(t)) dx \\ &\quad + 2n \alpha'(t) \int_0^1 [\partial_x d(\rho(t))]^{2n-1} \partial_x D(\rho(t)) dx. \end{aligned}$$

Apply Young's inequality to the second term on the right hand side to bound it by the sum of two terms. The first one can be absorbed by the first term on the right hand side, and the second one is bounded by $C_0(n^2/\nu)\alpha'(t)^2 F_{n-1}(t)$. In the

last term on the right hand side, replace $\partial_x D(\rho(t))$ by $\partial_x [D(\rho(t)) - D(\alpha(t))]$ and integrate by parts to obtain that it is equal to

$$-2n(2n-1) \alpha'(t) \int_0^1 [\partial_x d(\rho(t))]^{2(n-1)} \partial_x^2 d(\rho(t)) [D(\rho(t)) - D(\alpha(t))] dx .$$

Apply Young's inequality to bound this expression by the sum of two terms. The first one can be absorbed by the first term on the penultimate formula, while the second one is less than or equal to $C_0 n^2 \alpha'(t)^2 \nu^{-1} F_{n-1}(t)$. Therefore,

$$F'_n(t) \leq -c_0 n^2 \nu G_{n-1}(t) + C_0 n^2 \frac{\alpha'(t)^2}{\nu} F_{n-1}(t) .$$

Let $f(x) = [\partial_x d(\rho(t, x))]^n$. Since $\int_0^1 \partial_x d(\rho(t, x)) dx = 0$, there exists $x_0 \in [0, 1]$ such that $\partial_x d(\rho(t, x_0)) = 0$, so that $f(x_0) = 0$. We may therefore apply Poincaré's inequality to $[\partial_x d(\rho(t, x))]^n$ to obtain that

$$F_n(t) = \int f(x)^2 dx \leq K_1 \int f'(x)^2 dx = K_1 n^2 G_{n-1}(t) .$$

It follows from the previous estimates that

$$F'_n(t) \leq -b_0 \nu F_n(t) - c_0 n^2 \nu G_{n-1}(t) + C_0 n^2 \frac{\alpha'(t)^2}{\nu} F_{n-1}(t)$$

for some $b_0 > 0$. The same inequality remains in force for any $0 < b < b_0$. It remains to apply Gronwall inequality to complete the proof. \square

Iterating the inequality appearing in the previous assertion without the term G_{n-1} yields

Assertion 5.D. *For all $n \geq 2$, there exist positive constants $0 < c_0 < C_0 < \infty$, $b_0 > 0$, such that for all $0 < b < b_0$, $t \geq 0$,*

$$F_n(t) + c_0 n^2 \nu \int_0^t e^{-b\nu(t-s)} G_{n-1}(s) ds \leq r_n(t) ,$$

where $r_n(t) = r_n(t, b, C_0)$ is given by

$$\begin{aligned} r_n(t) &= C_0 \sum_{k=2}^n F_k(0) \left(\frac{t[n\beta_1(t)]^2}{\nu} \right)^{n-k} \frac{e^{-b\nu t}}{(n-k)!} \\ &+ C_0 \left(\frac{[n\beta_1(t)]^2}{\nu} \right)^{n-1} \int_0^t e^{-b\nu(t-s)} \frac{(t-s)^{n-2}}{(n-2)!} F_1(s) ds . \end{aligned}$$

Let $w : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ be given by

$$w(t, x) = [\partial_x^2 d(\rho_t)](x) - \frac{1}{\nu} \alpha'(t) . \quad (5.5)$$

Assertion 5.E. *There exist positive constants $0 < c_0 < C_0 < \infty$ such that*

$$\int_0^1 w(t)^2 dx \leq e^{-c_0 \nu t} \int_0^1 w(0)^2 dx + \frac{C_0}{\nu^3} \int_0^t \alpha''(s)^2 ds + r_2(t) ,$$

where the remainder r_2 has been introduced in Assertion 5.D.

Proof. The proof is similar to the one of Assertion 5.A. We first show that

$$\begin{aligned} \frac{d}{dt} \int_0^1 w(t)^2 dx &\leq -\nu \int_0^1 D(\rho_t) [\partial_x^3 d(\rho_t)]^2 dx + \frac{1}{A\nu^3} \alpha''(t)^2 \\ &\quad + \nu \int_0^1 \frac{1}{D(\rho_t)} [\partial_x D(\rho_t)]^2 [\partial_x^2 d(\rho_t)]^2 dx + A\nu \int_0^1 w(t)^2 dx \end{aligned}$$

for any $A > 0$. As $w(t)$ vanishes at $x = 0$, apply Poincaré's to this functions to get that

$$\int_0^1 w(t)^2 dx \leq K_1 \int_0^1 [\partial_x^3 d(\rho_t)]^2 dx .$$

Hence, choosing A small enough yields

$$\frac{d}{dt} \int_0^1 w(t)^2 dx \leq -c_0\nu \int_0^1 w(t)^2 dx + \frac{C_0}{\nu^3} \alpha''(t)^2 + C_0\nu G_1(t) ,$$

where the function G_1 has been introduced in (5.4). We may replace the constant c_0 by one which is smaller than the constant b_0 appearing in the statement of Assertion 5.D. By Gronwall inequality,

$$\int_0^1 w(t)^2 dx \leq e^{-c_0\nu t} \int_0^1 w(0)^2 dx + C_0 \int_0^t e^{-c_0\nu(t-s)} \left\{ \frac{\alpha''(s)^2}{\nu^3} + \nu G_1(s) \right\} ds .$$

To complete the proof of the lemma, it remains to recall the statement of Assertion 5.D. \square

Lemma 5.4. *Assume that $c_1\nu > 2K_1C_1\beta_1(t_0)$ for some $t_0 > 0$. Then, there exist positive constants $0 < c_0 < C_0 < \infty$ such that for all $0 \leq t \leq t_0$,*

$$\begin{aligned} \|\partial_x d(\rho(t))\|_\infty^2 &\leq C_0 e^{-c_0\nu t} \left\{ \int_0^1 w(0)^2 dx + F_2(0) + \frac{1}{\nu} e^{C_0\beta_1(t_0)} t F_1(0) \right\} \\ &\quad + \frac{C_0}{\nu^2} \left\{ \alpha'(t)^2 + \frac{1}{\nu} \int_0^t \alpha''(s)^2 ds + \frac{1}{\nu^2} e^{C_0\beta_1(t_0)[1+t]} \right\} . \end{aligned}$$

Proof. Assume that $2K_1C_1\beta_1(t_0) < c_1\nu$ for some $t_0 > 0$ and fix $0 < t \leq t_0$. Since $\int_0^1 (\partial_x d(\rho(t))) dx = 0$, subtracting this integral and applying Schwarz inequality, we get that for all $x \in [0, 1]$,

$$[\partial_x d(\rho(t, x))]^2 \leq C_0 \int_0^1 [\partial_x^2 d(\rho(t, y))]^2 dy .$$

Adding and subtracting $\alpha'(t)/\nu$, by Young's inequality, the previous expression is less than or equal to

$$C_0 \int_0^1 w(t)^2 dy + C_0 \frac{\alpha'(t)^2}{\nu^2} .$$

By Assertion 5.E, the first term of the previous expression is less than or equal to

$$C_0 e^{-c_0\nu t} \int_0^1 w(0)^2 dx + \frac{C_0}{\nu^3} \int_0^t \alpha''(s)^2 ds + r_2(t) .$$

By its definition and by Assertion 5.B,

$$r_2(t) \leq C_0 e^{-c_0\nu t} \left\{ F_2(0) + \frac{1}{\nu} e^{C_0\beta_1(t_0)} t F_1(0) \right\} + \frac{C_0}{\nu^4} e^{C_0\beta_1(t_0)[1+t]} .$$

To complete the proof it remains to recollect all previous estimates. \square

Let

$$q(t) = \|\partial_x d(\rho_t)\|_\infty, \quad Q(t) = \sup_{0 \leq s \leq t} q(s), \quad t \geq 0. \quad (5.6)$$

Assertion 5.F. *Suppose that $Q(t_0)(1 + n^2 K_1) < c_1$ for some $t_0 > 0$. Then, there exists a positive constant $C_0 < \infty$ such that for all $0 \leq t \leq t_0$,*

$$\int_0^1 w(t)^{2n} dx \leq e^{-a_n \nu t} \int_0^1 w(0)^{2n} dx + \int_0^t e^{-a_n \nu(t-s)} H_{n-1}(s) ds,$$

where $a_n = [c_1 - Q(t_0)(1 + n^2 K_1)]/K_1$ and

$$H_{n-1}(s) = C_0 \left\{ n^2 q(s)^2 \frac{\alpha'(s)^2}{\nu} + \frac{\alpha''(s)^2}{\nu^3} \right\} \int_0^1 w(s)^{2(n-1)} dx.$$

Proof. Since $w(t)$ vanishes at the boundary, an integration by parts yields that the time derivative of $\int_0^1 w(t)^{2n} dx$ is equal to

$$\begin{aligned} & -2n(2n-1)\nu \int_0^1 w(t)^{2(n-1)} D(\rho_t) [\partial_x^3 d(\rho_t)]^2 dx \\ & -2n(2n-1)\nu \int_0^1 w(t)^{2(n-1)} \partial_x D(\rho_t) w(t) \partial_x^3 d(\rho_t) dx \\ & -2n(2n-1)\alpha'(t) \int_0^1 w(t)^{2(n-1)} \partial_x D(\rho_t) \partial_x^3 d(\rho_t) dx - \frac{2n\alpha''(t)}{\nu} \int_0^1 w(t)^{2n-1} dx. \end{aligned} \quad (5.7)$$

In this formula, in the second line, we added and subtracted $(1/\nu)\alpha'(t)$ to recover $w(t)$ from $\partial_x^2 d(\rho_t)$.

Recall the definition of $q(t)$, introduced in (5.6), and the one of the constants c_1 , C_1 , defined in (5.3). Estimating $\partial_x D(\rho_t)$ by $C_1 q(t)$, and applying Young inequality to the last three terms of the previous displayed equation, we obtain that the time derivative of $\int_0^1 w(t)^{2n} dx$ is bounded by

$$\begin{aligned} & -2n(2n-1)\nu \left\{ c_1 - \frac{q(t)}{2} - \frac{1}{A} \right\} \int_0^1 w(t)^{2(n-1)} [\partial_x^3 d(\rho_t)]^2 dx \\ & + 2n(2n-1)\nu \left\{ \frac{q(t)}{2} + \frac{1}{A} \right\} \int_0^1 w(t)^{2n} dx \\ & + C_0 \left\{ n^2 A q(t)^2 \frac{\alpha'(t)^2}{\nu} + \frac{\alpha''(t)^2}{\nu^3} \right\} \int_0^1 w(t)^{2(n-1)} dx, \end{aligned}$$

for every $A > 0$.

Let $f(x) = w(t, x)^n$. Since f vanishes at the boundary and since $f'(x) = n w(t, x)^{n-1} \partial_x^3 d(\rho_t)$, by Poincaré's inequality,

$$\int_0^1 w(t)^{2n} dx = \int_0^1 f^2 dx \leq K_1 \int_0^1 [f']^2 dx = K_1 n^2 \int_0^1 w(t)^{2(n-1)} [\partial_x^3 d(\rho_t)]^2 dx.$$

Set $A = 2(1 + n^2 K_1)/c_1$. With this choice, by assumption, $c_1 - q(t)/2 - 1/A > 0$. In particular, the sum of the first two line of the penultimate displayed equation is less than or equal to

$$-\frac{2n(2n-1)\nu}{2n^2 K_1} \left\{ c_1 - q(t)(1 + n^2 K_1) \right\} \int_0^1 w(t)^{2n} dx.$$

Since $2n(2n-1)/2n^2 \geq 1$, to complete the proof it remains to apply Gronwall inequality. \square

Assertion 5.G. Assume that $2(1+K_1)C_1Q(t_0) < c_1$ for some $t_0 > 0$. Then, there exist positive constants $0 < c_0 < C_0 < \infty$ such that for all $0 \leq t \leq t_0$,

$$\int_0^1 [\partial_x^3 d(\rho_t)]^2 dx \leq e^{-a\nu t} \int_0^1 [\partial_x^3 d(\rho_0)]^2 dx + \int_0^t e^{-a\nu(t-s)} H(s) ds,$$

where $a = [c_1 - 2(1+K_1)C_1Q(t_0)]/2K_1$, and

$$H(s) = C_0 \nu q(s)^2 \int_0^1 [\partial_x^2 d(\rho_s)]^2 dx + C_0 \nu \int_0^1 [\partial_x^2 d(\rho_s)]^4 dx + \frac{C_0}{\nu^3} [\alpha''(s)]^2.$$

Proof. The proof is similar to the one of Assertion 5.B. Fix $t_0 > 0$ satisfying the hypothesis of the lemma and consider some $t < t_0$. Since $\rho(t, 1) = \alpha_t$, $\alpha''(t) = \nu^2 \partial_x^2 [D(\rho(t)) \partial_x^2 d(\rho(t))]$ at $x = 0, 1$. Adding and subtracting $\nu^{-1} \alpha''(t)$, and integrating by parts, we have that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 [\partial_x^3 d(\rho_t)]^2 dx &= -\nu \int_0^1 \partial_x^4 d(\rho_t) \partial_x^2 \{D(\rho_t) \partial_x^2 d(\rho_t)\} dx \\ &\quad + \nu^{-1} \alpha''(t) \int_0^1 \partial_x^4 d(\rho_t) dx. \end{aligned} \quad (5.8)$$

Let $D_1(\alpha) = (\log D)'(\alpha)$, $D_2(\alpha) = (\log D)''(\alpha)/D(\alpha)$. Expand $\partial_x^2 \{D(\rho_t) \partial_x^2 d(\rho_t)\}$, and observe that $\partial_x^2 D(\rho_t) = D_1(\rho_t) \partial_x^2 d(\rho_t) + D_2(\rho_t) [\partial_x d(\rho_t)]^2$ to write the first term on the right hand side of the previous formula as

$$\begin{aligned} &-\nu \int_0^1 D(\rho_t) [\partial_x^4 d(\rho_t)]^2 dx - 2\nu \int_0^1 \partial_x D(\rho_t) \partial_x^3 d(\rho_t) \partial_x^4 d(\rho_t) dx \\ &-\nu \int_0^1 D_1(\rho_t) [\partial_x^2 d(\rho_t)]^2 \partial_x^4 d(\rho_t) dx - \nu \int_0^1 D_2(\rho_t) [\partial_x d(\rho_t)]^2 \partial_x^2 d(\rho_t) \partial_x^4 d(\rho_t) dx. \end{aligned}$$

Recall the definition of $q(t)$ introduced in (5.6), and recall that $c_1 = \inf_{0 \leq \alpha \leq 1} D(\alpha)$, $C_1 = \|D_1\|_\infty$. Apply Young's inequality to the last three terms and to the last term in (5.8) to obtain that the left hand side of (5.8) is less than or equal to

$$\begin{aligned} &-\nu [c_1 - \frac{2}{A} - C_1 q(t)] \int_0^1 [\partial_x^4 d(\rho_t)]^2 dx + C_1 \nu q(t) \int_0^1 [\partial_x^3 d(\rho_t)]^2 dx \\ &+ A C_0 \nu q(t)^2 \int_0^1 [\partial_x^2 d(\rho_t)]^2 dx + A C_0 \nu \int_0^1 [\partial_x^2 d(\rho_t)]^4 dx + \frac{A}{\nu^3} [\alpha''(t)]^2 \end{aligned} \quad (5.9)$$

for all $A > 0$.

Since $\partial_x^2 d(\rho(t, 1)) = \partial_x^2 d(\rho(t, 0))$, $\int_0^1 \partial_x^3 d(\rho_t) dx = 0$. Therefore, by Poincaré's inequality,

$$\int_0^1 [\partial_x^3 d(\rho_t)]^2 dx \leq K_1 \int_0^1 [\partial_x^4 d(\rho_t)]^2 dx.$$

Set $A = 4/c_1$. Since, by hypothesis, $2C_1 q(t) \leq 2C_1 Q(t_0) < c_1$, the first line of (5.9) is bounded by

$$-\frac{\nu}{2K_1} [c_1 - 2C_1 [1 + K_1] q(t)] \int_0^1 [\partial_x^3 d(\rho_t)]^2 dx \leq -a \nu \int_0^1 [\partial_x^3 d(\rho_t)]^2 dx,$$

where a has been introduced in the statement of the assertion. To complete the proof, it remains to apply Gronwall inequality. \square

Proof of Proposition 5.1. The claims are straightforward consequences of Lemmas 5.3 and 5.4. We turn to the third assertion. \square

Proof of Proposition 5.2. Since $\partial_x^2 d(\rho)(t, 1) - (1/\nu)\alpha'(t)$ vanish as $x = 0$, by Schwarz inequality, for any $x_0 \in [0, 1]$,

$$[\partial_x^2 d(\rho_t)(x_0) - \alpha'(t)]^2 \leq \int_0^1 [\partial_x^3 d(\rho_t)(x)]^2 dx.$$

Fix $t_0 > 0$. By Proposition 5.1, $Q(t_0)^2 \leq C_0 \delta^2$, where $\delta^2 = \varepsilon^2 + \nu^{-2}$. Therefore, there exist $\varepsilon_0 > 0$ and $\nu_0 < \infty$ with the property that the hypothesis of Assertion 5.G is in force for all $\varepsilon < \varepsilon_0$, $\nu > \nu_0$. In particular, the previous expression is bounded by

$$C_0 \varepsilon^2 + \int_0^t e^{-a\nu(t-s)} H(s) ds,$$

where H has been introduced in the statement of Assertion 5.G. By Proposition 5.1, which permits to estimate $q(s)^2$, by adding and subtracting $\alpha'(s)$ to $\partial_x^2 d(\rho_s)$, which permits to recover the function $w(s)$ introduced in (5.5), the second term of the previous equation is less than or equal to

$$C_0 \left\{ \frac{1}{\nu^4} + \frac{\varepsilon^2}{\nu^2} \right\} + C_0 \nu \int_0^t ds e^{-a\nu(t-s)} \int_0^1 \{ \delta^2 w(s)^2 + [\partial_x^2 d(\rho_s)]^4 \} dx.$$

By Assertion 5.E, this sum is bounded by

$$C_0 \left\{ \frac{1}{\nu^4} + \varepsilon^4 \right\} + C_0 \nu \int_0^t ds e^{-a\nu(t-s)} \left\{ \delta^2 r_2(s) + \int_0^1 [\partial_x^2 d(\rho_s)]^4 dx \right\},$$

By Assertion 5.B, $r_2(s) \leq C_0 \delta^2$. We may thus remove r_2 from the previous formula. By Young inequality, by Assertion 5.F and by Proposition 5.1, the second term without $r_2(s)$ is less than or equal to

$$C_0 \left\{ \frac{1}{\nu^4} + \varepsilon^4 \right\} + C_0 \delta^2 \int_0^t ds e^{-a\nu(t-s)} \int_0^s dr e^{-a'\nu(s-r)} \int_0^1 w(r)^2 dx.$$

By Assertions 5.E and 5.B, the previous expression is less than or equal to $C_0 \delta^4$. This concludes the proof of the proposition. \square

Proof of Proposition 2.3. By Proposition 5.1, for every $t_0 > 0$, there exists $\nu_0 < \infty$ and $B < \infty$, where B depends on $\alpha(s)$, $0 \leq s \leq t_0$, and on the initial condition v_0 , such that for all $0 \leq t \leq t_0$

$$\|u_\nu(t)\|_\infty^2 \leq B, \quad \int_0^1 [\partial_x u_\nu(t)]^2 dx \leq B \quad (5.10)$$

Fix $t_0 > 0$ and $0 \leq t < t_0$. By definition, $u_\nu(t, 0) = u_\nu(t, 1) = 0$, and

$$\partial_t u_\nu = \nu \left\{ \partial_x [D(\alpha_t + \varepsilon u_\nu) \partial_x u_\nu] - \alpha'(t) \right\} = \nu \partial_x \left\{ D(\alpha_t + \varepsilon u_\nu) \partial_x u_\nu - D(\alpha_t) \partial_x v_t \right\}.$$

where $\varepsilon = \nu^{-1}$.

Therefore, for every $t \geq 0$, an integration by parts yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 [u_\nu(t) - v_t]^2 dx &= \int_0^1 [u_\nu(t) - v_t] \partial_t v_t dx \\ &\quad - \nu \int_0^1 \partial_x [u_\nu(t) - v_t] \{ D(\alpha_t + \varepsilon u_\nu) \partial_x u_\nu - D(\alpha_t) \partial_x v_t \} dx. \end{aligned}$$

The second term is less than or equal to

$$\begin{aligned} & -\nu \int_0^1 D(\alpha_t) [\partial_x u_\nu(t) - \partial_x v_t]^2 dx + C_0 \int_0^1 |\partial_x u_\nu(t) - \partial_x v_t| |u_\nu| |\partial_x u_\nu| dx \\ & \leq -c_0 \nu \int_0^1 [\partial_x u_\nu(t) - \partial_x v_t]^2 dx + \frac{C_0}{\nu} \int_0^1 u_\nu(t)^2 [\partial_x u_\nu(t)]^2 dx . \end{aligned}$$

By (5.10), the second term is bounded by B/ν . Therefore, by Young's inequality and by Poincaré's inequality,

$$\frac{1}{2} \frac{d}{dt} \int_0^1 [u_\nu(t) - v_t]^2 dx \leq -c_0 \nu \int_0^1 [u_\nu(t) - v_t]^2 dx + \frac{C_0}{\nu} \int_0^1 (\partial_t v_t)^2 dx + \frac{B}{\nu} .$$

To conclude the proof, it remains to apply Gronwall inequality. \square

6. THE DIFFUSION COEFFICIENT

We provide in this section of formula for the diffusion coefficient.

Fix a cylinder function $f : \{0, 1\}^{\mathbb{Z}} \rightarrow \mathbb{R}$, and recall from (3.17) that $\hat{f} : [0, 1] \rightarrow \mathbb{R}$ represents the polynomial defined by

$$\hat{f}(\theta) = E_{\nu_\theta} [f(\xi)] .$$

Write $E_{\nu_{\theta+h}} [f(\xi)]$ as $E_{\nu_\theta} [f(\xi) N_h]$, where N_h is the Radon-Nikodym derivative of $\nu_{\theta+h}$, restricted to the support of f , with respect to ν_θ , to get that

$$\hat{f}'(\theta) = \frac{1}{\mathfrak{c}(\theta)} \sum_{k \in \mathbb{Z}} \langle f; \eta(k) \rangle_\theta , \quad (6.1)$$

where $\langle f; g \rangle_\theta$ represents the covariance between two cylinder functions f, g in $L^2(\nu_\theta)$: $\langle f; g \rangle_\theta = E_{\nu_\theta} [fg] - E_{\nu_\theta} [f] E_{\nu_\theta} [g]$, and $\mathfrak{c}(\theta)$ the static compressibility, given by $\mathfrak{c}(\theta) = \theta(1 - \theta)$.

Recall the definitions of the cylinder function h , introduced in (3.15). We claim that

$$\hat{h}'(\theta) = D(\theta) . \quad (6.2)$$

Indeed, since the cylinder function $c(\eta)$ does not depend on $\eta(0)$ and $\eta(1)$,

$$\mathfrak{c}(\theta) D(\theta) = - \sum_{k \in \mathbb{Z}} k \langle [\eta(0) - \eta(1)] c(\eta); \eta(k) \rangle_\theta .$$

Note that all terms in this sum vanish but the one with $k = 1$, and that the sum over k is finite because c is a cylinder function. By (2.1) and by a change of variables, the right hand side is equal to

$$- \sum_{k \in \mathbb{Z}} k \sum_{a=1}^m \sum_{j \in \mathbb{Z}} \mu_a(j) \langle \tau_{-j} h_a; \eta(k) \rangle_\theta = - \sum_{a=1}^m \sum_{k, j \in \mathbb{Z}} k \mu_a(j) \langle \tau_{-(j+k)} h_a; \eta(0) \rangle_\theta .$$

Note that sum over j is finite because μ_a has finite support. By definition of m_a , and since the total mass of μ_a vanishes, $\sum_j \mu_a(j) = 0$, performing the change of variables $k' = j + k$ last term becomes

$$\sum_{a=1}^m m_a \sum_{k, \in \mathbb{Z}} \langle \tau_{-k} h_a; \eta(0) \rangle_\theta = \sum_{a=1}^m m_a \sum_{k, \in \mathbb{Z}} \langle h_a; \eta(k) \rangle_\theta = \mathfrak{c}(\theta) \sum_{a=1}^m m_a \hat{h}'_a(\theta) ,$$

where the last identity follows from (6.1). This last expression is equal to $\mathfrak{c}(\theta) \hat{h}'(\theta)$, which concludes the proof of (6.2).

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